UC IRVINE DISCUSSION SESSION

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Abstract. The purpose of this discussion session is to introduce some fundamental concepts in probability theory, and to derive the kinetic formulation of Burger's equation.

Probability

(1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent coin flips that satisfy
$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2.$$
Define the simple random walk $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ for every $n \in \mathbb{N}$. Prove that, $\mathbb{P}$-almost surely,
$$\lim_{n\to\infty} \frac{S_n}{n} = 0.$$
(Hint: Consider the fourth moment $\mathbb{E}[n^{-4}S_n^4]$ and apply the Borell–Cantelli lemma.)

(2) What happens if the coin flips are not fair: for some $p \in (0,1)$,
$$\mathbb{P}[X_i = 1] = p \;\text{ and }\; \mathbb{P}[X_i = -1] = 1 - p?$$

(3) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X\colon \Omega \to \mathbb{R}$ be a real-valued random variable. The distribution of X on $\mathbb{R}$ is the measure μ_X defined by
$$\mu_X(A) = \mathbb{P}[X^{-1}(A)] = \mathbb{P}[\{\omega \in \Omega\colon X(\omega) \in A\}] \;\text{ for every }\; A \in \mathcal{B}(\mathbb{R}).$$
The characteristic function of a random variable X is defined by
$$\phi_X(t) = \mathbb{E}[e^{itX}] = \int_R e^{itx}\mu_X(\,\mathrm{d}x).$$
Show that the distribution of X is uniquely determined by its characteristic function. That is, if $\phi_X = \phi_Y$ if and only if $\mu_X = \mu_Y$. (Hint: Find a connection with the Fourier transform.)

(4) Prove the central limit theorem: as $n \to \infty$,
$$\mathbb{P}[a \le S_n/\sqrt{n} \le b] \to \int_a^b (2\pi)^{-\frac{1}{2}} \exp(-|x|^2/2t).$$
(Hint: Prove the convergence of the characteristic functions, and apply Levy's continuity theorem.)

The Kinetic Formulation

(5) Consider the viscous Burger's equation, for $\eta \in (0,1)$,
$$\partial_t\rho_\eta + \frac{1}{2}\partial_x(\rho_\eta^2) = \eta\Delta\rho_\eta \;\text{ in }\; \mathbb{T}^1 \times (0,\infty) \;\text{ with }\; \rho(x,0) = \rho_0,$$
for a smooth, bounded ρ_0. Argue that the above equation possesses a smooth and bounded solution. (Hint: For example, use a fixed point argument.)

(6) For a smooth, convex function $S\colon \mathbb{R}\to\mathbb{R}$ derive the equation satisfied by the composition $S(\rho)$: for every $\psi\in \mathrm{C}^\infty(\mathbb{T}^d)$ show that

$$\partial_t\int_{\mathbb{T}^1} S(\rho_\eta)\psi(x) = -\eta\int_{\mathbb{T}^1} S'(\rho_\eta)\nabla\psi\cdot\nabla\rho_\eta - \eta\int_{\mathbb{T}^1} S''(\rho_\eta)\psi\left|\nabla\rho_\eta\right|^2 + \frac{1}{2}\int_{\mathbb{T}^1}\rho_\eta^2 S'(\rho_\eta)\nabla\psi + \frac{1}{2}\int_{\mathbb{T}^1}\rho_\eta^2\nabla\rho_\eta S''(\rho_\eta)\psi.$$

(7) Introduce an additional variable $\xi\in\mathbb{R}$ and let the kinetic function χ_η of ρ_η be defined by

$$\chi_\eta(x,\xi,\theta) = \mathbf{1}_{\{0<\xi<\rho_\eta(x,t)\}} - \mathbf{1}_{\{\rho_\eta(x,t)<\xi<0\}}.$$

Derive distributional equalities for the derivatives

$$\nabla_x\chi_\eta \text{ and } \partial_\xi\chi_\eta,$$

and show that

$$S(\rho_\eta(x,t)) = \int_{\mathbb{R}}\chi_\eta(x,\xi,t)\,\mathrm{d}\xi.$$

(8) Show using the above distributional equalities that

$$\partial_t\int_{\mathbb{T}^1}\int_{\mathbb{R}} S'(\xi)\psi(x)\chi_\eta = -\eta\int_{\mathbb{T}^1}\int_{\mathbb{R}} S'(\xi)\nabla\psi\cdot\nabla\chi_\eta - \int_{\mathbb{T}^1}\int_{\mathbb{R}} S''(\xi)\psi\,\mathrm{d}q_\eta - \frac{1}{2}\int_{\mathbb{T}^1}\int_{\mathbb{R}}\partial_\xi\chi_\eta\xi^2 S'(\xi)\nabla\psi + \frac{1}{2}\int_{\mathbb{T}^1}\int_{\mathbb{R}}\nabla_x\chi_\eta\xi^2 S''(\xi)\psi$$

for the measure $q_\eta = \eta\delta_0(\xi-\rho_\eta(x,t))\left|\nabla\rho_\eta\right|^2$ and therefore that, for every $\Psi\in \mathrm{C}_c^\infty(\mathbb{T}^1\times\mathbb{R})$,

$$\partial_t\int_{\mathbb{T}^1}\int_{\mathbb{R}}\Psi(x,\xi)\chi_\eta = -\eta\int_{\mathbb{T}^1}\int_{\mathbb{R}}\nabla_x\Psi\cdot\nabla\chi_\eta - \int_{\mathbb{T}^1}\int_{\mathbb{R}}\partial_\xi\Psi\,\mathrm{d}q_\eta - \frac{1}{2}\int_{\mathbb{T}^1}\int_{\mathbb{R}}\partial_\xi\chi_\eta\xi^2\nabla_x\Psi + \frac{1}{2}\int_{\mathbb{T}^1}\int_{\mathbb{R}}\nabla_x\chi_\eta\xi^2\partial_\xi\Psi.$$

And that the final term my be rewritten in the form

$$\partial_t\int_{\mathbb{T}^1}\int_{\mathbb{R}}\Psi(x,\xi)\chi_\eta = -\eta\int_{\mathbb{T}^1}\int_{\mathbb{R}}\nabla_x\Psi\cdot\nabla\chi_\eta - \int_{\mathbb{T}^1}\int_{\mathbb{R}}\partial_\xi\Psi\,\mathrm{d}q_\eta + \int_{\mathbb{T}^1}\int_{\mathbb{R}}\partial_\xi\chi_\eta\xi\nabla_x\Psi.$$

(9) Argue formally that you may pass to the limit $\eta\to 0$, for which $\rho_\eta\to\rho$ and $\chi_\eta\to\chi$ strongly, for χ the kinetic function of ρ. Argue that there exists a finite, nonnegative measure q on $\mathbb{T}^1\times\mathbb{R}$ such that, for every compactly supported $\Psi\in \mathrm{C}_c^\infty(\mathbb{T}^1\times\mathbb{R})$,

$$\partial_t\int_{\mathbb{T}^1}\int_{\mathbb{R}}\Psi(x,\xi)\chi_\eta = -\int_{\mathbb{T}^1}\int_{\mathbb{R}}\partial_\xi\Psi\,\mathrm{d}q + \int_{\mathbb{T}^1}\int_{\mathbb{R}}\partial_\xi\chi\xi\nabla_x\Psi.$$

(10) Show that, for solutions ρ_1 and ρ_2 with kinetic functions χ_1 and χ_2,

$$\int_{\mathbb{T}^1}\left|\rho_1-\rho_2\right| = \int_{\mathbb{T}^1}\int_{\mathbb{R}}\left|\chi_1-\chi_2\right|^2 = \int_{\mathbb{T}^1}\int_{\mathbb{R}}\chi_1\mathrm{sgn}(\xi)+\chi_2\mathrm{sgn}(\xi)-2\chi_1\chi_2.$$

Formally differentiate the above inequality to show that

$$\max_{t\in[0,T]}\left\|\rho_1-\rho_2\right\|_{L^1(\mathbb{T}^1)} \leq \left\|\rho_{0,1}-\rho_{0,2}\right\|_{L^1(\mathbb{T}^1)}.$$

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