THE ALT-PHILLIPS FREE BOUNDARY PROBLEM

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1. Introduction

Two of the most basic free boundary problems are the Obstacle Problem and the One-phase Bernoulli FBP. They both consist in minimizing an energy functional of the type

$$J(u, \Omega) := \int_\Omega \frac{|\nabla u|^2}{2} + W(u) dx,$$

among functions which are fixed on the boundary of Ω,

$$u = \varphi \geq 0 \quad \text{on} \quad \partial\Omega.$$

The function (potential) W is given, and it is assumed to be nonnegative and that it achieves its minimum at 0, i.e. $W : \mathbb{R} \to [0, \infty)$ and

$$W(t) = 0 \qquad \text{if} \quad t \leq 0, \qquad W(t) > 0 \qquad \text{if} \quad t > 0.$$

The first term in the energy is minimized by the harmonic function with boundary data φ while the second by $u = 0$. The presence of the potential term W has the effect of penalizing the positive values of u. Depending on the behavior of W near 0, minimizers can develop zero patches $u = 0$ inside the domain, in which case we have a free boundary problem.

In the Obstacle Problem the potential is $W(t) = t^+$. This corresponds to the physical situation of an elastic membrane at rest on top of a table. The membrane is represented by the graph of u, and it touches the table when $u = 0$ and it separates away from it when $u > 0$. The potential term W encodes the gravitational force acting on the membrane.

In the One-phase FBP the potential is $W(t) = \chi_{\{t>0\}}$. This corresponds to the physical situation when the positivity set of u is penalized uniformly, and then the minimizer u is harmonic in $\{u > 0\}$. A physical motivation for this problem comes from the study two-dimensional steady fluid flows, where u represents the stream lines of the flow and the region $\{u > 0\}$ the location of the fluid.

These basic two FBP have many features in common but also differences. They can be viewed as important special cases for the more general one-phase *Alt-Phillips FBP* given by the family of power potentials

$$W(t) = t^\gamma \quad \text{if} \quad t > 0, \quad \text{and} \quad \gamma \in (-2, 2).$$

Such potentials are relevant in applications in porous catalysts [A] and population dynamics [GM].

We note that the restriction of the exponent γ to the interval $(-2, 2)$ is necessary in order for the zero patches to exist. This can be seen from some simple 1D analysis. For example, if $\gamma \geq 2$, then $u > 0$ in Ω by the strong maximum principle, while if $\gamma \leq -2$, then the energy of a function with nontrivial zero set is infinite.

The study of this family of FBP was initiated by Phillips in [P] and Alt-Phillips in [AP] where the basic local regularity properties of minimizers and their free boundaries was establishedfor exponents $\gamma \in (0,2)$. The case of negative exponents was considered more recently in [DS2].

In this notes we present some of the main tools and ideas that are used in the study of the free boundary regularity for this family of problems with a focus on the classical cases $\gamma = 1$ and $\gamma = 0$.

2. General results for $\gamma \geq 0$

One-dimensional discussion. Assume that $u : [0,\delta] \to \mathbb{R}^+$ solves the ODE

$$u'' = \gamma u^{\gamma - 1} \quad \text{in } (0,\delta), \text{ and } u(0) = 0. \tag{2.1}$$

We multiply the equation by u' and integrate, and deduce that in $(0,\delta)$

$$\frac{1}{2} u'^2 = u^\gamma + \mu, \tag{2.2}$$

for some constant $\mu = \frac{1}{2}(u'(0+))^2 \geq 0$. We can compute u explicitly as $u = G^{-1}$ with

$$G : [0,\infty) \to [0,\infty), \qquad G(s) := \sqrt{2} \int_0^s (\mu + t^\gamma)^{-\frac{1}{2}} \, dt.$$

Assume further that the extension of u by 0 on the negative axis is a minimizer of J in the interval $[-\delta,\delta]$. Then we claim that $\mu = 0$ and $u = u_0$ where u_0 is the explicit function

$$u_0 := c_0 (t^+)^\alpha, \qquad \alpha = \frac{2}{2-\gamma}, \tag{2.3}$$

with c_0 the constant so that (2.1) holds. We call u_0 the *one-dimensional* solution. In other words in 1D, the minimality of the energy implies the equipartition of the energy.

For the claim we compare u with infinitesimal dilations with the same boundary data

$$u_\lambda(t) := u(\delta + \lambda(t - \delta))$$

and λ close to 1. Then

$$J(u_\lambda, [-\delta,\delta]) = \int_{-\delta}^{\delta} \left(\frac{\lambda}{2}(u')^2 + \frac{1}{\lambda} u^\gamma \right) dt$$

is minimal when $\lambda = 1$ which means $(u')^2/2 = u^\gamma$, and that gives $\mu = 0$.

Alternatively, we could use Cauchy-Schwartz inequality and write

$$J(u, [-\delta,\delta]) \geq \sqrt{2} \int_{-\delta}^{\delta} u^{\gamma/2} u' dt,$$

and the right hand side depends only on the values of u at the end points. The equality occurs when $\frac{1}{2}(u')^2 = u^\gamma$, i.e. $\mu = 0$.

This discussion applies for all values of $\gamma \in (-2,2)$. In particular, minimality implies $u'(0) = 0$ if $\gamma > 0$ and $u'(0) = \sqrt{2}$ if $\gamma = 0$.

Convexity of the energy. The value $\gamma = 1$ is borderline for the convexity of the energy. If $\gamma \geq 1$ then J is convex, which implies that critical points of J are minimizers and are unique. If $\gamma < 1$ then critical points are not necessarily minimizing (or stable) and uniqueness might fail. For example $c_0|t|^\alpha$ is a minimizer when $\gamma \geq 1$ but it is not when $\gamma \leq 1$.

Similarly, when $\gamma < 1$, the critical point

$$c_0[(t-1)^+]^\alpha + c_0[(t+1)^-]^\alpha$$

is minimizing in an interval $[-a, a]$ with $a > 1$ close to 1, however it is not minimizing (but only stable) if a is large.

Scaling. The equation

$$\triangle u = \gamma u^{\gamma - 1}$$

and the energy J remain invariant under the α-homogenous rescaling

$$\tilde{u}(x) = r^{-\alpha} u(rx),$$

with α as in (2.3). More generally, if we divide u by $a > 0$ and scale it by a $\frac{1}{r}$ dilation in x

$$\tilde{u}(x) = \frac{1}{a} u(rx), \tag{2.4}$$

it has the effect of multiplying the potential W by the constant $r^2 a^{\gamma-2}$, i.e. $\tilde{u}$ minimizes the energy

$$\tilde{J}(v, \tilde{\Omega}) = \int_{\tilde{\Omega}} \frac{|\nabla v|^2}{2} + \left(\frac{a}{r^\alpha}\right)^{\gamma - 2} \cdot W(v)\, dx, \qquad \tilde{\Omega} = \frac{1}{r}\Omega.$$

Continuity. We assume that the boundary data $\varphi \geq 0$ is the trace of a $H^1(\Omega)$ function. The existence of a minimizer $u \in H^1(\Omega)$ follows easily by the standard methods. We want to prove that u is continuous and that it is smooth in $\{u > 0\}$.

Note that we cannot write the Euler-Lagrange equation in the weak sense right away for $\gamma \in [0, 1)$ since we do not know yet that $W'(u)$ is integrable.

First we claim that u is subharmonic. Indeed, using that $W' \geq 0$ we have that for any $\psi \geq 0$, $\psi \in C_0^\infty(\Omega)$,

$$0 \leq J(u - \varepsilon\psi, \Omega) - J(u, \Omega) \leq \frac{1}{2} \int_\Omega |\nabla(u - \varepsilon\psi)|^2 - \nabla u^2\, dx,$$

hence, after letting $\varepsilon \to 0^+$ we find

$$\int \nabla u \cdot \nabla \psi \geq 0.$$

By the mean value property, u is bounded and we can define it pointwise as an upper semicontinuous function.

Lemma 2.1 (Harnack inequality)**.** *Let u be a minimizer in B_1, and assume that $u(0) \geq C$, with C large, universal. Then*

$$c_0 \cdot u(0) \leq u \leq C_0 \cdot u(0) \quad \text{in} \quad B_{1/2}.$$

Proof. If $\gamma > 1$, then

$$W'(u) \in L^\infty \quad \Longrightarrow \quad \triangle u \in L^\infty \quad \Longrightarrow \quad u \in C^1 \quad \Longrightarrow \quad W'(u) \in C^\beta.$$

The Euler-Lagrange equation is satisfied in the classical sense and the conclusion follows from the inequalities

$$0 \leq \triangle u \leq C(1 + u), \qquad u \geq 0,$$

and the standard Harnack inequality.

Assume that $\gamma \le 1$. Denote by a the average of u on ∂B_1,

$$a := \fint_{\partial B_1} u \ \ge u(0) \ge C_1,$$

and we claim that

$$\fint_{\partial B_{1/2}} u \ge a(1 - C a^{\frac{\gamma}{2}-1}).$$

Indeed, let $\tilde{u} = u/a$ be the rescaling of u as in (2.4) with $r = 1$.

Let h be the harmonic replacement of $\tilde{u}$ in B_1, hence

$$h(0) = 1, \quad W(h) \le C(1+h) \quad \Longrightarrow \quad \int_{B_1} W(h) dx \le C.$$

Minimality implies

$$\tilde{J}(\tilde{u}, B_1) \le \tilde{J}(h, B_1) \Longrightarrow \quad \frac{1}{2}\int_{B_1} |\nabla(\tilde{u} - h)|^2 \le \int_{B_1} a^{\gamma-2} W(h) dx \le C a^{\gamma-2},$$

hence

$$\fint_{\partial B_{1/2}} |\tilde{u} - h| \le C \|u - h\|_{H^1(B_1)} \le C a^{\frac{\gamma}{2}-1},$$

which gives the claim. Now we can iterate the conclusion and obtain

$$u(0) \ge \frac{a}{2}. \tag{2.5}$$

Indeed, by scaling, if a_k denotes the average of u on ∂B_r, with $r = 2^{-k}$ then we have proved

$$a_k \ge C_1 r^\alpha \quad \Longrightarrow \quad a_{k+1} \ge a_k \left(1 - C r a_k^{\frac{\gamma}{2}-1}\right).$$

This implies (2.5) as we let $k \to \infty$, provided that C_1 is chosen sufficiently large. Finally, since u is subharmonic we obtain

$$u \le h \cdot a \le C u(0) \quad \text{in} \quad B_{1/2},$$

from which the lower bound can be inferred as well.

□

Optimal regularity. Lemma 2.1 implies that u is continuous and the sets $\{u = 0\}$, $\{u > 0\}$ and the free boundary

$$\Gamma := \partial\{u > 0\},$$

are well defined in a pointwise sense. Given a point $x_0 \in \{u > 0\}$, we rescale u by the α-homogenous rescaling

$$\tilde{u}(x) = r^{-\alpha} u(x_0 + rx),$$

so that $\tilde{u}(0) = 1$. Then Harnack inequality implies that $u \in [c_0, C_0]$ in a ball B_c with c small universal provided that $B_{2c} \subset \tilde{\Omega}$. As a consequence we obtain the following bound on the growth of u away from Γ.

Lemma 2.2 (Optimal growth)**.**

$$u(x) \le C \left(d_\Gamma(x)\right)^\alpha, \qquad \textit{if} \quad B_{2 d_\Gamma}(x) \subset \Omega,$$

with $d_\Gamma(x)$ representing the distance function to the set Γ.

In a similar fashion we have

Proposition 2.3 (Optimal regularity)**.** *Assume u is a minimizer in B_1 and $0 \in \Gamma$. Then*

$$\|u\|_{C^\alpha(B_{1/2}} \leq C.$$

When α is an integer, the C^α norm represents the usual $C^{\alpha-1,1}$ norm. If the set $\{u = 0\}$ does not intersect say $B_{3/4}$, then the right hand side should be replaced by $C\|u\|_{L^\infty(B_1)}$.

Exercise 1: Use interior estimates and Lemma 2.1 to prove Proposition 2.3.

Nondegeneracy. Next we show that a minimizer growth as $|x|^\alpha$ in some direction away from Γ.

Lemma 2.4 (Nondegeneracy)**.** *Assume $0 \in \Gamma$. Then*

$$\max_{\partial B_r} u \geq c r^\alpha, \qquad \forall r \leq 1.$$

Proof. By scaling, it suffices to show the statement for $r = 1$. Define

$$\psi(x) := \delta[(|x| - \frac{1}{2})^+]^{\max\{2,\alpha\}},$$

with δ, small universal and assume by contradiction that $u < \psi$ on ∂B_1. We compare the energies of u and ψ in the set $\{u > \psi\} \subset\subset B_1$ and obtain:

$$0 \geq \int_{\{v>0\}} \frac{1}{2}|\nabla v|^2 - v \cdot \triangle\psi + W(\psi + v) - W(\psi)\, dx, \qquad v := (u - \psi)^+.$$

The integrand is positive since

$$\triangle\psi \leq C\delta \quad \text{if } \gamma \leq 1, \text{ or } \quad \triangle\psi \leq C\delta W'(\psi) \quad \text{if } \gamma > 1,$$

and we reach a contradiction. □

We also state the compactness property of minimizers whose proof is left as an exercise.

Proposition 2.5 (Compactness of minimizers)**.** *Let u_k be a sequence of minimizers of J in B_1 which converges uniformly to u locally in B_1.*

Then, u is a minimizer in B_1. Moreover, on compact sets, the free boundaries $\Gamma(u_k)$ converge to $\Gamma(u)$ in the Huasdorff distance sense.

Exercise 2: Prove Proposition 2.5.

Monotonicity formula.

Theorem 2.6 (Weiss monotonicity formula)**.** *Let u be a minimizer to J in B_R then*

$$W_u(r) := r^{-n-2(\alpha-1)} J(u, B_r) - \frac{\alpha}{2} r^{-(n-1)-2\alpha} \int_{\partial B_r} u^2 d\sigma, \quad 0 < r \leq R,$$

is increasing in r. Moreover, W_u is constant if and only if u is homogeneous of degree α.

Notice that the optimal growth implies that if $0 \in F(u)$ then $W_u(r)$ is bounded below as $r \to 0$.

Proof. The quantity $W_u(r)$ is differentiable for a.e. r and by standard computations

$$\frac{d}{dr}J(u,B_r)=\int_{\partial B_r}(\frac{1}{2}|\nabla u|^2+W(u))dx,$$

while

$$\frac{d}{dr}\left(r^{-(n-1)-2\alpha}\int_{\partial B_r}u^2\right)d\sigma=2r^{-n-2\alpha}\int_{\partial B_r}(ruu_\nu-\alpha u^2)d\sigma.$$

Assume that these equalities are satisfied at $r=1$. Then,

$$\frac{dW_u}{dr}|_{r=1}=\int_{\partial B_1}(\frac{1}{2}|\nabla u|^2+W(u))d\sigma-(n-2+2\alpha)J(u,B_1)$$
$$-\alpha\int_{\partial B_1}(uu_\nu-\alpha u^2)d\sigma,$$

from which we deduce

$$\frac{dW_u}{dr}|_{r=1}=\int_{\partial B_1}(\frac{1}{2}u_\tau^2+W(u))d\sigma+\frac{\alpha^2}{2}\int_{\partial B_1}u^2d\sigma-(n-2+2\alpha)J(u,B_1)$$
$$+\frac{1}{2}\int_{\partial B_1}(u_\nu-\alpha u)^2d\sigma.$$

We claim that

$$I(u):=\int_{\partial B_1}(\frac{1}{2}u_\tau^2+W(u))d\sigma+\frac{\alpha^2}{2}\int_{\partial B_1}u^2d\sigma\geq(n-2+2\alpha)J(u,B_1).$$

Indeed let

$$\tilde{u}(x):=|x|^\alpha u\left(\frac{x}{|x|}\right),\quad x\in B_1$$

the α-homogeneous extension of $u|_{\partial B_1}$. Then

$$I(u)=I(\tilde{u})=(n+1+2\alpha)J(\tilde{u},B_1),$$

where the last equality follows from the computation above for $\frac{d}{dr}W_{\tilde{u}}$ which is 0. and our claim follows from minimality. Thus,

$$\frac{d}{dr}W_u(r)\geq 0,\quad \text{a.e. } r.$$

The conclusion follows since $W_u(r)$ is absolutely continuous in r. Moreover, the computations above show that W_u is constant if and only if

$$u_\nu=\frac{\alpha}{|x|}u,\quad \text{for a.e. } x,$$

that is u is homogeneous of degree α.

□

Exercise 3: Instead of minimality, use that u is a critical point with respect to domain variations and show the equality

$$\frac{d}{dr}W_u(r)=r^{-n-2\alpha}\int_{\partial B_r}(ru_\nu-\alpha u)^2d\sigma.$$

Blow-ups. As a consequence we obtain that if $0\in\Gamma$, then as we let $r\to 0$, the rescalings

$$u_r(x):=r^{-\alpha}u(rx),$$

converge uniformly on compacts sets (along subsequences $r_k \to 0$) to a nonzero global homogenous minimizer, i.e. *a minimizing cone.*

In order to understand the local regularity properties of Γ near 0 it remains is to investigate the uniqueness of the blow-up limit and the classification of minimizing cones at least in low dimensions.

For a cone, the value W_u is constant, and it represents the energy of the cone.

We state a dimension reduction property of minimizers. From this one can deduce that the cone of least energy must be one-dimensional.

Lemma 2.7. *Assume u is constant in the e_1 direction, i.e.,*

$$u(x_1, \dots, x_n) = v(x_2, \dots, x_n).$$

Then u is a minimizing cone in $\mathbb{R}^n$ if and only if v is a minimizing cone in $\mathbb{R}^{n-1}$.

Exercise 4: Prove Lemma 2.7.

3. The Obstacle Problem

In this section we classify all cones in the Obstacle Problem ($\gamma = 1$).

Convexity. An important observation is that all cones must be convex.

Lemma 3.1. *Let u be a cone in the Obstacle Problem. Then u is convex.*

Proof. The second derivatives are bounded, homogenous of degree zero harmonic functions in the cone $\{u > 0\}$. Heuristically, the pure second derivatives u_{ee} are nonnegative at the boundary of the cone, and by the maximum principle they cannot have a negative minimum which must occur in the interior of the cone.

Rigorously, assume by contradiction that

$$\inf_{\{u>0\}} u_{ee} = m < 0.$$

Let $x_k \in \{u > 0\}$ be a sequence of points such that $u_{ee}(x_k) \to m$, and let $r_k = d_\Gamma(x_k)$. Rescale u so that $B_{r_k}(x_k)$ is mapped into B_1, and denote the rescaled function by u_k. Then, up to subsequences, u_k converges to a limit global solution $\bar{u}$. The infimum value of $\bar{u}_{ee}$ in $\{\bar{u} > 0\}$ is achieved at the origin, and by maximum principle, $\bar{u}_{ee}$ must be constant in the connected component containing 0. This means that $\bar{u}$ is a concave second order polynomial on the line te, $t \in \mathbb{R}$, and this contradicts the C^1 continuity where it becomes 0.

$\square$

Exercise 5: Show that cones are convex for all $\gamma \in [1, 2)$.
(Apply the maximum principle to $D_{ee} u^{\frac{2}{\alpha}}$.)

Boundary Harnack. Next we show that if u is a cone in the Obstacle Problem then $\{u = 0\}$ is either a half-space, or a lower dimensional subspace.

Lemma 3.2. *If $\{u = 0\}$ has nonempty interior then u is the one-dimensional solution up to rotations, i.e. $u = u_0(x)$ with*

$$u_0 := \frac{1}{2}(x_n^+)^2.$$

If $\{u=0\}$ has empty interior then u is a quadratic polynomial, i.e. $u=p(x)$,

$$p=\frac{1}{2}x^TAx, \qquad \text{with } A\geq 0,\ tr\,A=1.$$

The one-dimensional cone has least energy. All the quadratic cones have double the energy of the one-dimensional cone.

Proof. If $\{u=0\}$ contains the direction $-e$, the convexity implies that $u_e\geq 0$. If $-e$ is interior to $\{u=0\}$ then $u_e>0$ in $\{u>0\}$, and Γ is a locally Lipschitz graph in the e direction.

Each derivative u_ξ is a harmonic homogenous of degree one-function which vanishes on Γ. This means that on the unit sphere $\mathcal{S}^{n-1}$, u_ξ is an eigenfunction with eigenvalue $n-1$ for the spherical Laplacian $\triangle_{\mathcal{S}}$ in the Lipschitz domain $\{u>0\}\cap\mathcal{S}^{n-1}$. Since $u_e>0$, then it is a first eigenfunction. This means that each u_ξ is a multiple of u_e which implies that u is one-dimensional.

On the other hand, if $\{u=0\}$ has empty interior, then $\triangle u=1$ a.e. in $\mathbb{R}^n$. Since $u\in C^{1,1}$ this equation is satisfied in the weak sense, thus $u\in C^2$ and the equation holds everywhere. The desired conclusion follows from the Liouville theorem.

□

Exercise 6: Let u be a cone for some exponent $\gamma\in[1,2)$. Show that either u is one-dimensional, or it is of cylindrical type - the product of a cone with isolated singularity in $\mathbb{R}^k$ times $\mathbb{R}^{n-k}$.

Uniqueness of blow-ups. We show that if a solution u is well approximated by the one dimensional cone u_0 in B_1, then it is well approximated by slight rotations of u_0 in all balls B_r, $r>0$. We present a proof that is different than the original proof of Caffarelli.

Proposition 3.3 (Improvement of flatness)**.** *Assume that u is a solution to the Obstacle Problem in B_1 and*

$$|u-u_0|\leq\varepsilon \quad in \quad B_1,$$

with $\varepsilon\leq\varepsilon_0$ small universal. Then

$$|u-\bar u_0|\leq\frac{\varepsilon}{2}\rho^2 \quad in \quad B_\rho,$$

for some ρ universal, and $\bar u_0$ denotes a rigid motion perturbation of u_0:

$$\bar u_0(x)=\frac{1}{2}[(x\cdot\nu-a)^+]^2, \quad |\nu-e_n|\leq C\varepsilon, \quad |a|\leq C\varepsilon.$$

Proof. Nondegeneracy and optimal growth imply that

$$\Gamma\cap B_{1/2}\subset\{|x_n|\leq C\varepsilon^{1/2}\}.$$

Denote the rescaled error function by

$$v:=\frac{1}{\varepsilon}(u-u_0), \qquad |v|\leq 1.$$

A key observation is that $|v|$ is subharmonic since $(u-u_0)^+$ and $(u-u_0)^-$ are subharmonic.

Since in $B_{1/2}$, $v=0$ on $x_n=-C\varepsilon^{1/2}$, and $|v|\leq 1$, we conclude that

$$|v|\leq C'(x_n+C\varepsilon^{1/2}) \quad \text{in} \quad B_{1/4}.$$

On the other hand v is harmonic in $x_n \geq C\varepsilon^{1/2}$, hence

$$|v-h| \leq C_1 \varepsilon^{1/2} \quad \text{in} \quad B_{1/4} \cap \{x_n \geq C\varepsilon^{1/2}\},$$

where h denotes the harmonic function that vanishes on $\{x_n = C\varepsilon^{1/2}\} \cap B_{1/4}$ and agrees with v on the remaining portion of the boundary. We find that

$$|v - a x_n - b_i x_i x_n| \leq \frac{1}{4}\rho^2 \quad \text{in} \quad B_\rho,$$

from which the conclusion follows with ν the unit direction of the vector $e_n + \varepsilon b_i e_i$. □

By iterating Proposition 3.3 we obtain the $C^{1,\beta}$ regularity of Γ near the points that admit a one-dimensional blow-up profile. Such points are refereed to as *regular points*.

Corollary 3.4. *Under the hypotheses of Proposition 3.3, the free boundary Γ is a $C^{1,\beta}$ graph in the e_n direction with norm bounded by $C\varepsilon$.*

Higher regularity and analyticity of Γ at regular points follows from the work of Kinderlehrer-Nirenberg.

The uniqueness of the quadratic blow-up cones (at the *singular points*) follows from the Monneau monotonicity formula that we leave as an exercise.

Exercise 7. Assume that 0 is a singular point. Then

$$\frac{d}{dr} \fint_{\partial B_r} (u-p)^2 d\sigma \geq 0,$$

where p is a quadratic cone.

Cones for γ close to 1. The quadratic polynomials p in the obstacle problem represent a continuous family of cones. There is however a more rigid picture in the case when γ is close to 1, $\gamma \neq 1$. Recently with Hui Yu in [SY1] we showed that up to rotations only the axis symmetric quadratic polynomials i.e.

$$p_k = \frac{1}{2k} \sum_{i=1}^{k} x_i^2, \qquad \text{for some} \quad k \leq n,$$

can appear as limits of cones with exponents $\gamma_m \to 1$, $\gamma_m \neq 1$.

Open problem: Show that all cones for exponents $\gamma \neq 1$ close to 1, are axially symmetric, that is, they are the extension to $\mathbb{R}^n$ of a radial cone in $\mathbb{R}^k$ for some $k \leq n$.

We remark that in 2D the cones can be analyzed for all exponents γ by studying the corresponding nonlinear ODE on the unit circle. Bonorino et al. in [BBLT] computed the number of cones (up to rotations) explicitly in terms of γ. This number tends to ∞ as $\gamma \to 2$, and in particular it shows that as $\gamma > 1$ increases there are cones in 2D that are not radially symmetric.

4. The One-Phase Problem

In this section we discuss more in detail the One-Phase FBP ($\gamma = 0$). A major difference with the previous section is that the energy is no longer convex. Thus a solution that is minimizing the energy in small balls is not necessarily a minimizer (or stable) in the whole domain Ω. On the other hand, solutions are harmonic in their positivity set which makes it easier to deal with the interior equation.

We write the energy in its more customary form

$$J(u,\Omega) =) \int_\Omega |\nabla u|^2 + \chi_{\{u>0\}}\, dx,$$

so that the free boundary condition on the smooth part of Γ takes the form

$$|\nabla u| = 1 \quad \text{on} \quad \Gamma.$$

The problem remains invariant under the Lipschitz rescaling

$$u_r(x) = r^{-1} u(rx).$$

Two examples of global solutions that are not globally minimizing are

$$(|x_n| - 1)^+ \quad \text{and} \quad c_n \left(1 - |x|^{2-n}\right)^+,$$

and the family generated by their rescalings.

Viscosity solutions. It is convenient to have a more general notion of solution that is preserved when taking uniform limits of classical solutions.

Definition 4.1. A continuous function $u : \Omega \to \mathbb{R}^+$ is a *viscosity supersolution* to the One-Phase FBP if

a) $\triangle u \le 0$ in $\{u > 0\}$;

b) u cannot be touched by below at $x_0 \in \Gamma$ by a test function φ^+, that satisfies

$$\varphi \in C^1, \quad \varphi(x_0) \neq 0, \quad |\nabla \varphi(x_0)| > 1.$$

Similarly, u is a *viscosity subsolution* if

a') $\triangle u \ge 0$ in $\{u > 0\}$;

b') u cannot be touched by above at $x_0 \in \Gamma$ by a test function φ^+, that satisfies

$$\varphi \in C^1, \quad \varphi(x_0) \neq 0, \quad |\nabla \varphi(x_0)| < 1.$$

A *viscosity solution* satisfies both the subsolution and supersolution properties.

Notice that $|x_n|$ is a viscosity solution, and so is $a x_n^+ + b x_n^-$ with $a, b \in (0, 1]$.

Exercise 8: Show that if u minimizes the energy J in small balls, then u is a viscosity solution.

Optimal growth. Viscosity solutions grow at most linearly away from Γ.

Lemma 4.2. *Let u be a viscosity solution to the One-Phase FBP. Then*

$$u(x) \le C d_\Gamma(x),$$

provided that $B_{d_\Gamma(x)}(x) \subset \Omega$.

Proof. By scaling we may assume that $B_1 \subset \{u > 0\}$ is tangent to Γ at some point $x_0 \in \partial B_1$. It suffices to show that $u(0) \le C$.

Since $u > 0$ is harmonic in B_1 we find from Harnack inequality that

$$u \ge cu(0) \quad \text{in} \quad B_{1/2},$$

hence

$$u \ge c'u(0)(|x|^{2-n} - 1) \quad \text{in} \quad B_1 \setminus B_{1/2},$$

by the maximum principle. This contradicts the supersolution property at x_0 if $u(0)$ is sufficiently large. □

By using the optimal growth one can show that the free boundary has finite perimeter.

Exercise 9: Assume that u is a viscosity solution in B_1. Show that the level sets $\{u = \varepsilon\}$ for all small ε, have bounded $\mathcal{H}^{n-1}$ measure in $B_{1/2}$.
(Hint: Integrate $\triangle u$ in the set $\{u > \varepsilon\} \cap B_{1/2}$.)

Nondegeneracy. Minimizers satisfy a nondegeneracy condition that takes a stronger form than the one in the Obstacle Problem.

Lemma 4.3 (Nondegeneracy)**.** *Assume u is a minimizer of J. Then*

$$u(x) \ge c\, d_\Gamma(x),$$

provided that $B_{d_\Gamma(x)}(x) \subset \Omega$.

Proof. As above we may assume that $B_1 \subset \{u > 0\}$ and we need to show that $u(0) > c$. By Harnack inequality

$$u \le Cu(0) \quad \text{in} \quad B_{1/2}.$$

If $u(0)$ is sufficiently small, then

$$v := \min\left\{u, (r_0^{2-n} - |x|^{2-n})^+\right\}, \qquad r_0 = \frac{1}{2} - \delta,$$

coincides with u on $\partial B_{1/2}$. On the other hand v has less energy than u in $B_{1/2}$ since

$$J(u, B_{1/2}) \ge |B_{1/2}|, \qquad J(v, B_{1/2}) \le C\delta,$$

and we reach a contradiction. □

Density estimates. Another difference with the Obstacle Problem is that the zero set $\{u = 0\}$ of minimizers satisfies a uniform density estimate.

Lemma 4.4 (Uniform density)**.** *Let u be a minimizer of J. Assume that $0 \in \Gamma$. Then*

$$1 - c \ge \frac{|\{u = 0\} \cap B_r|}{|B_r|} \ge c, \quad \forall B_r \subset \Omega.$$

Proof. After rescaling we may assume $r = 1$.

By Lemma 2.4 and Lemma 4.3 we know that $\{u > 0\} \cap B_1$ contains a ball $B_c(x_0)$ for some c universal, and $u(x_0) \geq c_1$. This gives the upper bound of the density.

For the lower bound we denote by

$$a(r) := |\{u = 0\} \cap B_r|,$$

and let h be the harmonic replacement of u in B_1. The minimality implies

$$a(1) \geq \int_{B_1} |\nabla (h-u)|^2 dx \geq c \left(\int_{B_1} (h-u)^{2^*} dx \right)^{2/2^*}.$$

We have

$$h(x_0) \geq u(x_0) \geq c_1 \quad \Longrightarrow \quad h \geq c_2 (1 - |x|).$$

By integrating on the set $\{u = 0\}$ we find

$$a(1)^{1+\delta} \geq c_3 t^{C_0} \cdot a(1-t),$$

with $\delta = 2^*/2 - 1$, $C_0 = 2^*$ depending only on n.

We write this inequality for consecutive terms of the sequence

$$a_k := a(1/2 + 2^{-k}), \quad k \geq 1,$$

and find that if a_1 is sufficiently small, then $a_k \to 0$ as $k \to \infty$. This implies $a(1/2) = 0$ which gives $\{u > 0\}$ in $B_{1/2}$, and we reach a contradiction.

□

Improvement of flatness. Next we discuss the improvement of flatness for viscosity solutions following the approach of De Silva [D].

Proposition 4.5 (Improvement of flatness)**.** *Let u be a viscosity solution such that*

$$0 \in \Gamma \quad \text{and} \qquad (x_n - \varepsilon)^+ \leq u \leq (x_n + \varepsilon)^+ \quad \text{in} \quad B_1,$$

for some $\varepsilon \leq \varepsilon_0$ small. Then

$$(x \cdot \nu - \frac{\varepsilon}{2}\rho)^+ \leq u \leq (x \cdot \nu + \frac{\varepsilon}{2}\rho)^+ \quad \text{in} \quad B_\rho,$$

with $\rho > 0$ universal, and

$$|\nu - e_n| \leq C\varepsilon, \qquad |\nu| = 1.$$

Proof. We denote by v the rescaled error function in the set $\overline{\{u > 0\}}$

$$v := \frac{1}{\varepsilon}(u - x_n), \qquad |v| \leq 1.$$

We want to show that v is well approximated by the solution to the linearized problem, that is, the Laplace equation with Neumann boundary condition in B_1^+.

Clearly v is harmonic in $\{x_n > \varepsilon\} \cap B_1$, and $v(0) = 0$.

Step 1: (Harnack inequality) We show that the oscillation of v is decaying at a fixed rate when restricted to dyadic balls $B_r(x_0)$ centered at points

$$x_0 \in \{x_n = 0\} \cap B_{1/2}, \quad \text{as long as } r \geq C\varepsilon.$$

Assume for simplicity that $x_0 = 0$. The claim follows if we show that in $B_{1/2}$ one of the two original bounds improved i.e.

$$\text{either} \quad u \geq (x_n - (1-\delta)\varepsilon)^+, \qquad \text{or} \quad u \leq (x_n + (1-\delta)\varepsilon)^+,$$

for some $\delta > 0$ universal. Then we can iterate this conclusion in dyadic balls till the radius of the ball becomes comparable to ε.

For the improvement on the original bounds, let's assume that at the point $\bar{x} = \frac{1}{4}e_n$, u is closer to the top constraint than to the bottom one, that is, $u \geq x_n$ at $\bar{x}$. Then Harnack inequality implies that

$$u - (x_n - \varepsilon) \geq c\varepsilon \quad \text{in} \quad B_{1/8}(\bar{x}).$$

Then we use small vertical translations of order ε of the explicit barrier

$$\varphi(x) := x_n - \varepsilon + c_1\varepsilon(|x - \bar{x}|^{2-n} - (3/4)^{2-n}) \quad \text{in} \quad B_{3/4}(\bar{x}) \setminus B_{1/8}(\bar{x}),$$

which are admissible as test functions for the supersolution property. We reach the conclusion that $u \geq \varphi^+$, which gives the desired improvement on the lower bound.

Step 2; (Compactness) We consider a sequence of error functions v_k corresponding to a sequence of $\varepsilon_k \to 0$ (and functions u_k), and show that for k large enough v_k is approximated by a solution to the Neumann problem.

By Step 1, we can extract a subsequence of the v_k's whose graphs converge uniformly on compact sets of the cylinder $\overline{B}_{1/2} \times \mathbb{R}$ to the graph of a Hölder function $\bar{v}$ that satisfies

$$\bar{v} : \overline{B}^+_{1/2} \to [-1, 1], \qquad \triangle \bar{v} = 0, \qquad \bar{v}(0) = 0.$$

We check that $\bar{v}$ satisfies the Neumann condition

$$\bar{v}_n = 0 \quad \text{on} \quad \{x_n = 0\},$$

in the viscosity sense.

Assume by contradiction that $\bar{v}$ is touched strictly by below at 0 in a small neighborhood by a quadratic polynomial P that satisfies

$$P_n(0) > 0, \quad \triangle P > 0.$$

Then a small vertical translation of

$$\varphi_k(x) := x_n + \varepsilon_k P$$

touchs u_k by below at a point that converges to 0 as $k \to \infty$. On the other hand, φ_k is an admissible test functions for the supersolution property and we reach a contradiction.

The estimates for $\bar{v}$ imply

$$|\bar{v} - b' \cdot x'| \leq \frac{1}{4}\rho \quad \text{in} \quad B_\rho,$$

from which the conclusion follows.

□

As in the previous section, Proposition 4.5 implies that Γ is smooth in a neighborhood of a *regular point*, that is a point where the blow-up limit cone is one-dimensional.

The linearized equation. Next we assume that Γ is smooth and u is a classical solution to the One-Phase problem in Ω, and we deduce its linearized equation.

We write a generic nearby solution in the form

$$(u + \varepsilon v)^+,$$

where u is extended away from $\overline{\{u>0\}}$ in a C^2 fashion. We require that the function above solves the problem up to first order in ε. The interior equation gives

$$\triangle v = 0 \quad \text{in} \quad \{u>0\}.$$

In order to check the boundary condition for v at some point $x_0 \in \Gamma$, it is convenient to write a perturbed function as above in a neighborhood of x_0 as a domain perturbation of u:

$$u(x+\varepsilon \Phi), \qquad \Phi = v\, \frac{\nabla u}{|\nabla u|^2}.$$

The choice of Φ is so that the two functions differ by $O(\varepsilon^2)$. Then, we have

$$1 = |\nabla u \cdot (I + \varepsilon D\Phi)| + O(\varepsilon^2), \quad \text{at} \quad x = x_0,$$

hence we find

$$\nabla u \cdot D\Phi \cdot (\nabla u)^T = 0 \qquad \text{at} \quad x_0.$$

We choose a system of coordinates where e_n is the inner normal to Γ at x_0 pointing towards the positivity set, hence $\nabla u(x_0) = e_n$, and obtain

$$\left(\frac{v u_n}{|\nabla u|^2} \right)_{x_n} = 0 \quad \Longrightarrow \quad v_n - u_{nn} v = 0 \qquad \text{at} \quad x_0.$$

Notice that $u_{ij}(x_0)$ with $i,j<n$ are the entries of the second fundamental form of Γ at x_0, thus

$$\triangle u = 0 \quad \Longrightarrow \quad u_{nn} = -H,$$

where H denotes the mean curvature of Γ at x_0 oriented towards the complement.

In conclusion, the linearized equation is

$$\begin{cases} \triangle v = 0 & \text{in } \{u>0\}, \\ \\ v_\nu + H\, v = 0 & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where ν denotes the inner normal to Γ and H the mean curvature of Γ.

Clearly, the derivatives u_e of u solve the Linearized equation (4.1).

The linearized equation has variational structure with the associated energy given by

$$\int_{\{u>0\}} |\nabla v|^2 dx - \int_\Gamma H v^2 \, d\sigma. \tag{4.2}$$

Stability. Next we discuss the stability of conical solutions which are classical outside the origin, that is, their free boundaries have smooth cross section on the unit sphere.

First we remark that $|\nabla u|$ is a 0-homogenous subharmonic function in $\{u>0\}$ which takes the value 1 on the boundary. By maximum principle

$$|\nabla u| \le 1 \quad \text{in} \quad \{u>0\} \quad \Longrightarrow \quad H = -u_{\nu\nu} \ge 0 \quad \text{on} \quad \Gamma.$$

A general principle states that the stability of a solution in a domain $\mathcal{U}$ is equivalent to the first eigenvalue of the linearized operator being nonnegative. This means that stability/instability of a solution is equivalent to the existence of positive/sign-changing solutions to the linearized equation.

In the case of cones, solutions to the linearized equation (4.1), can be obtained by separations of variables as

$$v = f(r)\bar{v}(\theta), \qquad r = |x|, \quad \theta = \frac{x}{|x|} \in \mathcal{S}^{n-1},$$

with

a) $\bar{v} > 0$ is the first eigenfunction of (4.1) on the unit sphere:

$$\begin{cases} \triangle_S \bar{v} = \Lambda \bar{v} & \text{in } \{u > 0\} \cap \mathcal{S}^{n-1}, \\ \bar{v}_\nu + H\,\bar{v} = 0 & \text{on } \Gamma \cap \mathcal{S}^{n-1}, \end{cases}$$

and $-\Lambda$ denotes the first eigenvalue.

b) f is a solution to the second order ODE

$$f'' + (n-1)\frac{f'}{r} f' + \Lambda \frac{f}{r^2} = 0.$$

The ODE has sign changing solutions if and only if

$$\Lambda > \left(\frac{n}{2} - 1\right)^2 \tag{4.3}$$

which is the sharp criteria for the instability of u.

In practice, such an inequality on the first eigenvalue can be shown by finding an explicit subsolution $\tilde{v} \geq 0$ on the unit sphere :

$$\begin{cases} \triangle_S \tilde{v} \geq \left(\frac{n}{2} - 1\right)^2 \tilde{v} & \text{in } \{u > 0\} \cap \mathcal{S}^{n-1}, \\ \tilde{v}_\nu + H\,\tilde{v} \geq 0 & \text{on } \Gamma \cap \mathcal{S}^{n-1}, \end{cases}$$

with the inequalities being strict at least at one point.

This can be written equivalently, in terms of the existence of a globally defined function v which is homogenous of a certain degree $-\mu$ and that is a subsolution to an appropriate eigenvalue problem.

Exercise 10: Inequality (4.3) is equivalent to the existence of a function $v \geq 0$, homogenous of degree $-\mu$, which is a strict subsolution for the following problem

$$\begin{cases} \triangle v \geq \left(\frac{n}{2} - 1 - \mu\right)^2 \frac{v}{|x|^2} & \text{in } \{u > 0\}, \\ v_\nu + H\,v \geq 0 & \text{on } \Gamma \setminus \{0\}. \end{cases} \tag{4.4}$$

The criteria (4.4) was used in Jerison-Savin [JS] to show that in dimension $n \leq 4$, the only stable cones are one-dimensional. This improved the prior results of Alt-Caffarelli [AC] for $n = 2$, and Caffarelli-Jerison-Kenig [CJK] for $n = 3$. As a consequence

Theorem 4.6 ([JS])**.** *Let u be a minimizer of J. Then Γ is smooth if $n \leq 4$.*

The idea in [JS] is to choose a convenient test subsolution

$$v = f(\lambda_1, .., \lambda_n),$$

with f a homogenous symmetric function of the eigenvalues of $D^2 u$. Some of the key computations are given in the next exercises.

Exercise 11: Let u be a harmonic function. Show that

$$|\nabla u|^{1-\frac{1}{n}} \quad \text{and} \quad |D^2 u|^{1-\frac{2}{n}} \quad \text{are subharmonic.}$$

Here we use the norm

$$|D^2 u| = |(\lambda_1, .., \lambda_n)| = (\Sigma\, u_{ij}^2)^{1/2}.$$

Exercise 12: Let u be a cone with smooth cross section. Show that $v = |D^2 u|^{\frac{1}{3}}$ satisfies (4.4) in dimension $n = 3$.

Axis symmetric cones. We consider cones that have x_n as axis of symmetry, and are invariant under rotations in the first $n-1$ variables. Denote by

$$t = x_n, \qquad s = |x'|,$$

and let u_{as} be the 1-homogenous harmonic function even in t that is obtained by rotating

$$u_{as}(x) = h(s,t) = r f(\theta),$$

along the t axis, where r, θ denote the polar coordinates in the s,t plane.

Up to a multiplicative constant, the function f satisfies the ODE

$$f'' - (n-2)\tan\theta \cdot f' + (n-1)\cdot f = 0, \qquad f(0) = 1, \quad f'(0) = 0,$$

and it is defined in the maximal interval $[-\theta_0, \theta_0]$ where f remains positive. Then, on the unit sphere H is constant

$$H = (f''/f')(\theta_0) = (n-2)\tan\theta_0.$$

According to the analysis above, to check the stability of this solution we need to solve the ODE

$$g'' - (n-2)\tan\theta \cdot g' + \left(n - 1 - (\frac{n}{2} - 1)^2\right)\cdot g = 0, \qquad g(0) = 1, \quad g'(0) = 0,$$

and then compare g'/g and $(n-2)\tan\theta$ at the end point of the interval θ_0.

The numerics have been carried out in [CJK, H], and it turns out that

$$u_{as} \text{ is unstable if } n \le 6 \text{ and stable if } n \ge 7.$$

However the instability can be deduced from a simple computation (due to X. Cabrè) by considering

$$v := |\nabla_{x'} u_{as}| = h_s \ge 0,$$

as a test subsolution. Indeed,

$$\triangle u_{as} = \triangle h + (n-2)\frac{h_s}{s} = 0,$$

and differentiating the second equation with respect to s, we obtain

$$\triangle v = (n-2)\frac{v}{s^2} \ge (n-2)\frac{v}{|x|^2}.$$

Notice that v is a subsolution on Γ since it is the supremum over a collection of derivatives of u, and v is homogenous of degree 0. From (4.4) we obtain that u_{as} is unstable if

$$n - 2 \ge \left(\frac{n}{2} - 1\right)^2 \quad \Longleftrightarrow \quad n \le 6.$$

De Silva and Jerison in [DJ] showed that u_{as} is in fact a minimizing cone in dimension $n = 7$ by constructing appropriate sub/supersolutions on both sides of the cone. The construction is nontrivial and requires some numerics. Dimension $n = 7$ is the least in which a nontrivial minimizing cone is known to exist.

These results are the counterpart of the work of Bombieri-De Giorgi-Giusti in the case of minimal surfaces.

Open Problem: Show that all minimizing cones are one dimensional in dimensions $n = 5, 6$ (or find a counterexample.)

5. Alt-Phillips as a degenerate One-Phase Problems

It is convenient to make a change of variables to a solution of the Alt-Phillips problem with general exponent γ so that the new function intersects the zero phase transversally as in the case $\gamma = 0$. In this way, at least in the case when the free boundary is a nice graph in the e_n direction, we can perform a Hodograph transformation and end up with a Lipschitz function in a fixed domain with flat boundary. This strategy is usually employed to establish the higher regularity.

The hodograph transformation referred above interchanges the x_n and x_{n+1} variables, and that has the effect of rotating the graph by 90°.

As observed by Alt and Phillips, after a simple change of variables

$$w = u^{1/\alpha}, \qquad \alpha := \frac{2}{2-\gamma}, \qquad \alpha \in (1, \infty),$$

the problem above can be viewed as a One Phase FBP for w. It turns out that w is Lipschitz and it solves a degenerate equation of the type

$$\Delta w = \frac{h(\nabla w)}{w} \qquad \text{in } \{w > 0\}, \tag{5.1}$$

with

$$\nabla w \subset \{h = 0\} \qquad \text{on } \partial\{w > 0\}, \tag{5.2}$$

where h is the quadratic polynomial

$$h(p) = \frac{\gamma}{\alpha} - (\alpha - 1)|p|^2.$$

A key feature of equation (5.1) is that it remains invariant under Lipschitz scaling $\tilde{w}(x) = w(rx)/r$. The right hand side degenerates as w approaches 0 and the free boundary condition (5.2) can be understood as a natural balancing condition in order to seek out for Lipschitz solutions w.

Viscosity solutions. More generally, we consider nonlinearities as in (5.1) with $h \in C^1(\mathbb{R}^n)$ that vanish on the boundary of a star shaped C^1 domain D. Assume that $0 \in D$ and

$$h \geq 0 \quad \text{in } D, \quad h \leq 0 \quad \text{in } \bar{D}^c, \tag{5.3}$$

$$h(p) \geq -C|p|^2, \quad C > 0, \quad \text{as } |p| \to \infty. \tag{5.4}$$

The Alt-Phillips problem with exponent γ corresponds to the particular case

$$h(p) = (\alpha - 1)(1 - |p|^2), \qquad D = B_1, \qquad \alpha - 1 = \frac{\gamma}{2-\gamma} \in [0, \infty). \tag{5.5}$$

The degenerate one-phase FBP for continuous function $w \geq 0$ becomes

$$
(5.6) \qquad \begin{cases} \Delta w = \dfrac{h(\nabla w)}{w} & \text{in } \Omega \cap \{w > 0\}, \\ \\ \nabla w \in \partial D, & \text{on } \Gamma(w) := \partial\{w > 0\} \cap \Omega. \end{cases}
$$

The equalities in (5.6) are understood in the viscosity sense. The notion of viscosity solution for the interior equation is standard, while the free boundary condition is as in the Definition 4.1, with the inequalities in b) and b') replaced by

$$\nabla \varphi \in \overline{D}^c \quad \text{and} \quad \nabla \varphi \in D.$$

As mentioned above, a key feature of the problem (5.6) is that it is invariant under Lipschitz rescalings

$$\tilde{w}(x) := \frac{w(rx)}{r}, \quad x \in B_1.$$

General results. The degenerate problem (5.6) was investigated in [DS1]. It turns out that many of the results from the one-phase problem continue to carry on. We only mention some of the main results in [DS1]:

a) existence of viscosity solutions by Perron's method;

b) optimal growth and regularity of solutions, i.e.

$$w(x) \leq C d_\Gamma(x), \quad \|w\|_{C^{0,1}} \leq C, \quad w \in C^{2,\beta}_{loc}(\{w > 0\});$$

c) nondegeneracy of the least viscosity solution;

d) $\mathcal{H}^{n-1}$ measure bound of the free boundary Γ for nondegenerate solutions;

e) improvement of flatness at flat free boundary points.

We make a few comments regarding the interior regularity for solutions to (5.1). This is not immediate as the right hand side degenerates either as $w \to 0$ or $\nabla w \to \infty$. It is useful to apply the work of Imbert and Silvestre [IS] where the weak Harnack estimates for solution to linear equations that are uniformly elliptic only when the gradient is large were developed. Notice that (5.3)-(5.4) imply that $\triangle w \leq 0$ when $|\nabla w|$ is large, while w^M is subharmonic for some large constant M. These are sufficient to establish a local Harnack inequality and obtain a uniform Hölder modulus of continuity for w.

The linearized equation. We comment on the type of linearized equations for these degenerate problems. Assume for simplicity that we are in the Alt-Phillips problem with exponent γ, i.e. h and D are as in (5.5), and assume that a solution w satisfies the flatness assumption

$$(x_n - \varepsilon)^+ \leq w \leq (x_n + \varepsilon)^+ \quad \text{in} \quad B_1.$$

It turns out that the rescaled function

$$\tilde{w} = \frac{w - x_n}{\varepsilon},$$

is well approximated by a solution of the following linearized equation:

$$
(5.7) \qquad \Delta v + s\frac{v_n}{x_n} = 0 \quad \text{in } B_1^+, \qquad s := 2(\alpha - 1) = \frac{2\gamma}{2 - \gamma},
$$

with the "Neumann" boundary condition

$$\partial_{x_n^{1-s}} v = 0 \quad \text{on } \{x_n = 0\}, \tag{5.8}$$

that is satisfied in the viscosity sense.

The value of the coefficient s plays an important role in the equation (5.7) and it is not allowed to be less than or equal to -1. Precisely, the boundary condition (5.8) is understood in the following sense:

a) If $s \geq 1$ then v is bounded near $\{x_n = 0\}$.

b) If $s \in (-1, 1)$ then v is continuous up to $\{x_n = 0\}$ and it cannot be touched by below (above) locally by the family of comparison functions

$$p(x') + t x_n^{1-s} \quad \text{with } t > 0 \text{ (or } t < 0) \text{ and } p(x') \text{ quadratic,}$$

at points on $\{x_n = 0\}$.

When $s \in (-1, 1)$, Problem (5.7) in $\mathbb{R}^{n+}$ is simply the extension problem of Caffarelli-Silvestre for the Dirichlet to Neumann operator representing

$$\Delta_{x'}^{\frac{1-s}{2}} v \quad \text{on} \quad \{x_n = 0\}.$$

In this range of s, both the Dirichlet and the Neumann boundary conditions can be imposed on $x_n = 0$. However, when $s \geq 1$, only the Neumann condition can be imposed and it simply requires the function to be bounded. When $s \leq -1$, the Dirichlet condition is meaningful, but the Neumann condition in the sense defined above cannot be imposed.

It is not too difficult to show that solutions to the Neumann problem (5.7)-(5.8) have pointwise $C^{1,\beta}$ estimates (see [DS1] for the details).

Theorem 5.1. *Assume that v is a solution of (5.7)-(5.8), and $s > -1$. Then*

$$|v(x) - v(0) - a' \cdot x'| \leq C \|v\|_{L^\infty} |x|^{1+\beta},$$

with C large, $\beta > 0$ small, depending only on n and s.

Axis symmetric cones for γ close to 1. With Hui Yu in [SY2] we studied the minimality of the radial and the axis symmetric cones for exponents $\gamma \in (1 - \delta, 1)$ with δ small. We showed that the radial cone is not minimizing in dimension $n = 2$ but it is minimizing in dimension $n = 3$.

More interestingly, we showed the existence of an axis symmetric minimizing cone in dimension $n = 4$ whose zero set has positive density. This is the analogue of the De Silva-Jerison cone. Thus in this range of exponents, there are minimizing cones with the features of the Obstacle Problem but also cones with the features of the One-Phase Problem.

Exercise 13: Assume that $\gamma < 1$ is close to 1. Show that the radial cone is stable in dimension $n = 3$ but unstable in dimension $n = 2$.

6. Negative exponents

In this section we consider the case of negative exponents

$$\gamma \in (-2, 0).$$

In order to figure out the correct free boundary condition for minimizers, we go back to the one-dimensional example (2.1)-(2.2). We see that the general solution to the ODE that vanishes at 0, and that is positive in $(0,\delta)$, has the form

$$u(t) = G^{-1}(t) = c_0 t^\alpha + \mu c_1 t^{2-\alpha} + O(t^\sigma), \tag{6.1}$$

with

$$\alpha = \frac{2}{2-\gamma} \quad \in (\frac{1}{2}, 1), \qquad \sigma > 2-\alpha,$$

and positive constants c_0, c_1 and σ depending only on γ.

The minimizing solution is the one with $\mu = 0$. This means that for negative exponents, the minimality condition imposes the value of the second coefficient in the expansion of a generic solution. We remark that for positive exponents this involved the value of the first coefficient in the expansion.

The second coefficient is not easily detected using integration, so it is convenient to use the notion of viscosity solution to define the free boundary condition. We show below that minimizers are indeed viscosity solutions.

Viscosity solutions. We consider the one-phase free boundary problem:

$$\begin{cases} \Delta u = -u^{\gamma-1} \quad \text{in } \{u>0\} \cap B_1, \\ u(x_0 + t\nu) = c_0 t^\alpha + o(t^{2-\alpha}) \quad \text{on } \Gamma(u) := \partial\{u>0\} \cap B_1, \end{cases} \tag{6.2}$$

with $t \ge 0$, ν the unit normal to $F(u)$ at x_0 pointing towards $\{u > 0\}$, and

$$\alpha := \frac{2}{2-\gamma}, \qquad c_0 := [\alpha(1-\alpha)]^{-\frac{1}{\gamma+2}}, \quad \gamma \in (-2,0), \quad \alpha \in (\frac{1}{2}, 1). \tag{6.3}$$

When we say that u touches $\phi \ge 0$ *strictly* by above at x_0, means that $u \ge \phi$ in a neighborhood B of x_0 and $u > \phi$ (except at x_0) in $B \cap \overline{\{\phi > 0\}}$. Similarly, *strictly* by below means the inequality to be strict in a neighborhood of x_0 intersected with $\overline{\{u > 0\}}$.

For test functions for the boundary condition, we only consider functions that depend on the distance to the boundaries of balls in $\mathbb{R}^n$:

a) denote by $\mathcal{C}^+$ the class of continuous functions ϕ for which there exists a ball

$$B := B_R(z_0), \qquad z_0 \in \mathbb{R}^n,$$

so that

$$\phi(x) = \phi(|x - z_0|) > 0 \quad \text{in} \quad B, \qquad \phi = 0 \quad \text{in} \quad B^c.$$

In this case we use the notation

$$d(x) := dist(x, \partial B) \quad \text{ if } x \in B, \text{ and } d(x) = 0 \text{ if } x \in B^c.$$

b) similarly we define the class $\mathcal{C}^-$, by interchanging B and B^c in part a).

Definition 6.1. We say that a non-negative continuous function u satisfies (6.2) in the viscosity sense, if

1) in the set where $u > 0$, u is C^∞ and satisfies the equation in a classical sense;

2) if $x_0 \in \Gamma$, then u cannot touch $\psi \in \mathcal{C}^+$ (resp. $\mathcal{C}^-$) by above (resp. below) at x_0, with

$$\psi(x) := c_0 d(x)^\alpha + \mu\, d(x)^{2-\alpha},$$

α, c_0 as in (6.3) and $\mu > 0$ (resp $\mu < 0$).

Exercise 14: Show that the barrier ψ in the definition above can be conveniently modified locally by adding to it a lower order term $d(x)^\sigma$ so that it becomes a subsolution (supersolution) of the interior equation (and in addition the touching may be assumed to be strict at x_0.)

Results. In [DS2] we investigated the properties of minimizers and established:
a) existence of minimizers
b) optimal regularity and nondegeneracy
c) improvement of flatness
d) $C^{1,\beta}$ regularity of Γ up to a closed singular set of codimension 3.
e) A Gamma convergence result to the perimeter functional as $\gamma \to -2$.

We state the Gamma convergence result for appropriate multiples of the J functional as $\gamma \to -2$. Let Ω be a bounded Lipschitz domain. We equip the space of nonnegative integrable functions

$$X := \{u \in L^1(\Omega), u \geq 0\}$$

with the distance

$$d_X(u,v) := \|u - v\|_{L^1} + \|\chi_{\{u>0\}} - \chi_{\{v>0\}}\|_{L^1}.$$

Theorem 6.2. *As $\gamma \to -2$, the rescaled J functionals*

$$\mathcal{J}_\gamma(u) := c_\gamma \, J(u, \Omega), \qquad c_\gamma := (1 + \frac{\gamma}{2})\sqrt{|\gamma|/2},$$

Gamma converge in X to the perimeter function

$$\mathcal{P}(u) = Per_\Omega(\{u > 0\}).$$

Precisely,
a) if $u_n \to u$ in X and $\gamma_n \to -2$, then $\liminf \mathcal{J}_{\gamma_n}(u_n) \geq \mathcal{P}(u)$;
b) given $u \in X$, there exists $u_n \to u$ in X such that $\mathcal{J}_{\gamma_n}(u_n) \to \mathcal{P}(u)$.

Minimizers as viscosity solutions. Finally we show how the second coefficient in the expansion appears in the minimality condition. We focus on the free boundary condition which is more delicate.

Lemma 6.3. *Let u minimize J in B_1. Then u is a viscosity solution to* (6.2).

Proof. Let us assume that u touches ψ by above at $x_0 \in F(u)$, with ψ as in Definition 6.1 and $\mu > 0$. Then in view of Exercise 14, u touches ϕ strictly by above at x_0, with ϕ defined as

$$\phi := c_0 d^\alpha + \frac{\mu}{2} d^{2-\alpha} + d^\sigma,$$

We will show that this contradicts the minimality of u, using a calibration argument. For simplicity, assume that the unit normal to $F(u)$ at x_0 is e_n. For any nonnegative function v, smooth in its positivity set, we denote by Γ_v its graph in $\mathbb{R}^{n+1}$ over the positivity set, and by $\nu_v(x)$ the upward unit normal to Γ_v at $(x, v(x))$.

Notice that we can write the energy of u over a domain Ω as a surface integral over its positivity graph in Ω, $\Gamma_u(\Omega)$, in the following way:

$$J(u, \Omega) = \int_{\Gamma_u(\Omega)} G(u, \nu_u) d\sigma, \tag{6.4}$$

with

$$G(s,\nu) := \frac{1}{2}\frac{|\nu'|^2}{\nu_{n+1}} + W(s)\nu_{n+1},$$

and

$$s>0, \quad |\nu| = 1, \quad \nu := \langle \nu', \nu_{n+1}\rangle, \quad \nu_{n+1} > 0.$$

Let $G(s,y)$ is the 1-homogeneous extension (in y) of $G(s,\nu)$. Then,

$$\nabla_y G(s,\nu) := \langle \frac{\nu'}{\nu_{n+1}}, -\frac{1}{2}\frac{|\nu'|^2}{\nu_{n+1}^2} + W(s)\rangle,$$

and the homogeneity and convexity in y imply,

$$G(\phi(x), \nu_\phi(x)) = V_\phi(x,\phi(x)) \cdot \nu_\phi(x), \tag{6.5}$$

with

$$V_\phi(x,\phi(x)) := \nabla_y G(\phi(x), \nu_\phi(x)),$$

and

$$G(\phi(x), \nu_u(x)) \geq V_\phi(x,\phi(x)) \cdot \nu_u(x). \tag{6.6}$$

The vector field $V_\phi(x,\phi(x))$ is defined on the graph Γ_ϕ, and we extended in $\mathbb{R}^{n+1}$ constantly in the e_n direction and denote it simply by V. This vector field is associated with the graphs of the translations

$$\phi_t(x) := \phi(x + te_n), \quad t \in \mathbb{R},$$

which provide a foliation of a neighborhood of $(x_0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^+$. In other words, for each given point $X := (x, x_{n+1})$ in this set, we identify the element ϕ_{t_X} of the foliation that passes through it i.e $x_{n+1} = \phi_{t_X}(x)$ and

$$V(X) = V_{\phi_{t_X}}(X).$$

Now, set

$$D := \{u(x) < x_{n+1} < \phi_{\bar{t}}(x)\} \quad \subset \mathbb{R}^{n+1},$$

with $\bar{t} > 0$ chosen in such a way that D is included in the neighborhood of $(x_0, 0)$ foliated by the graphs of the ϕ_t's. Denote by

$$D_\varepsilon := D \cap \{x_{n+1} > \varepsilon\}, \quad \text{and} \quad \Gamma_\varepsilon := D \cap \{x_{n+1} = \varepsilon\},$$

for $\varepsilon > 0$ small. Then, by the divergence theorem,

$$\int_{D_\varepsilon} div\, V\, dX = \int_{\Gamma_{\phi_{\bar{t}}} \cap \partial D_\varepsilon} V \cdot \nu_{\phi_{\bar{t}}}\, d\sigma - \int_{\Gamma_u \cap \partial D_\varepsilon} V \cdot \nu_u\, d\sigma - \int_{\Gamma_\varepsilon} V \cdot e_{n+1}\, dx,$$

and in view of (6.5)-(6.6),

$$\int_{D_\varepsilon} div\, V\, dX \geq \int_{\Gamma_{\phi_{\bar{t}}} \cap \partial D_\varepsilon} G(\phi_{\bar{t}}, \nu_{\phi_{\bar{t}}})\, d\sigma - \int_{\Gamma_u \cap \partial D_\varepsilon} G(u, \nu_u)\, d\sigma - \int_{\Gamma_\varepsilon} V \cdot e_{n+1}\, dx.$$

From the formula for V, on Γ_ε, for ε small,

$$V(x,\varepsilon) \cdot e_{n+1} = -\frac{1}{2}|\nabla \phi_{t_X}|^2 - \frac{1}{\gamma}\phi_{t_X}^\gamma \leq 0.$$

Indeed, we only need to verify that the one variable function of d,

$$\phi(d) := c_0 d^\alpha + \frac{\mu}{2} d^\beta + d^\sigma,$$

satisfies:

$$\frac{1}{2}\phi'^2 \geq \frac{1}{|\gamma|}\phi^\gamma, \quad \text{if } d > 0 \text{ is small.}$$

Since $\mu > 0$, we know that $\phi \geq c_0 d^\alpha$ while $\phi' \geq \alpha c_0 d^{\alpha-1}$. Hence, by the definition of α, c_0 (see (6.3)),

$$\phi'^2 \geq \alpha^2 c_0^2 d^{2(\alpha-1)} = \alpha^2 c_0^2 d^{\alpha\gamma} \geq \alpha^2 c_0^{2-\gamma} \phi^\gamma = \frac{2}{\gamma}\phi^{-\gamma},$$

as desired.

Finally, this implies that, after letting $\varepsilon \to 0$,

$$\int_D div\, V \; dX \geq \int_{\Gamma_{\phi_{\bar t}} \cap \partial D} G(\phi_{\bar t}, \nu_{\phi_{\bar t}})\; d\sigma - \int_{\Gamma_u \cap \partial D} G(u, \nu_u)\; d\sigma. \tag{6.7}$$

Next we show that

$$div\, V = -\triangle \phi_{t_X} - \phi_{t_X}^{\gamma-1} < 0,$$

and the left hand side in the inequality (6.7) is non-positive, which in view of the definition of G contradicts the minimality of u (see (6.4)).

To compute $div\, V$ at a point $(z_0, \phi(z_0))$, let

$$D_\varphi := \{0 < \phi(x) - \varepsilon\varphi(x) < x_{n+1} < \phi(x), x \in B_\delta(z_0)\}$$

with $\varphi(z_0) > 0$ and φ a smooth bump function supported on $B_\delta(z_0) \subset \{\varphi > 0\}$. Then, by a similar computation as above,

$$\int_{D_\varphi} div\, V \; dX = \int_{\Gamma_\phi} G(\phi, \nu_\phi)\; d\sigma - \int_{\Gamma_{\phi-\varepsilon\varphi}} G((\phi - \eta\varphi), \nu_{\phi-\varepsilon\varphi})\; d\sigma + O(\varepsilon^2),$$

where we used that if $x_{n+1} = \phi(x) - \varepsilon\varphi(x) = \phi_t(x)$, then $\nu_{\phi-\varepsilon\varphi}(x) = \nu_{\phi_t}(x) + O(\varepsilon)$ and by the homogeneity and C^2 smoothness of G

$$G(x_{n+1}, \nu_{\phi-\varepsilon\varphi}(x)) = \nabla_y G(x_{n+1}, \nu_{\phi_t}(x)) \cdot \nu_{\phi-\varepsilon\varphi}(x) + O(\varepsilon^2).$$

Thus, for ε small,

$$\begin{aligned}\int_{D_\varphi} div\, V \; dX =& \int_{B_\delta(z_0)} \left(\frac{1}{2}|\nabla\phi|^2 + W(\phi)\right) dx \\ &- \int_{B_\delta(z_0)} \left(\frac{1}{2}|\nabla(\phi - \varepsilon\varphi|^2 + W(\phi - \varepsilon\varphi)\right) dx + O(\varepsilon^2) \\ =& \varepsilon \int_{B_\delta(z_0)} \left(\nabla\phi \cdot \nabla\varphi - \phi^{\gamma-1}\varphi\right) dx + O(\varepsilon^2) \\ =& \varepsilon \int_{B_\delta(z_0)} \left(-\Delta\phi - \phi^{\gamma-1}\right) \varphi dx + O(\varepsilon^2).\end{aligned}$$

We divide by ε and let $\varepsilon \to 0$ and then $\delta \to 0$. Since $|D_\varphi| = \varepsilon \int \varphi dx$, and D_φ tends to $(z_0, \phi(z_0))$, we conclude that at $(z_0, \phi(z_0))$

$$div\; V = -\Delta\phi - \phi^{\gamma-1}.$$

The desired conclusion follows since ϕ is a subsolution.

□

Future directions. Some further directions for the study of the Alt-Phillips problem could be

a) the two-phase setting
b) developing the gradient flow theory
c) the p-Laplace setting
d) classification of global stable solutions in low dimensions

References

[AC] H. W. Alt, L. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., 325 (1981), 105-144.

[AP] H. W. Alt, D. Phillips, *A free boundary problem for semilinear elliptic equations*, J. Reine Angew. Math., 368 (1986), 63-107.

[A] R. Aris, *The mathematical theory of diffusion and reaction in permeable catalysts*, Oxford 1975.

[BBLT] Bonorino L., Brietzke E., Lukaszczyk J., C. Taschetto C., *Properties of the period function for some Hamiltonian systems and homogeneous solutions of a semilinear elliptic equation* Jour. Diff. Equations (2005) 156–175.

[CJK] L. Caffarelli, D. Jerison, C. Kenig, C, *Global energy minimizers for free boundary problems and full regularity in three dimensions*, Contempt. Math., 350 (2004), 83-97.

[D] D. De Silva, *Free boundary regularity for a problem with right hand side*, Interfaces Free Bound., 13 (2011), no. 2, 223-238.

[DJ] D. De Silva, D. Jerison, *A singular energy minimizing free boundary*, J. Reine Angew. Math., 635 (2009), 1-21.

[DS1] D. De Silva, O. Savin, *On certain degenerate one-phase free boundary problems*, SIAM J. Math. Anal., 53 (2021), no. 1, 649-680.

[DS2] D. De Silva, O. Savin, *The Alt-Philips functional for negative powers*, Bull. Lond. Math. Soc., 55 (2023), no. 6, 2749-2777.

[GM] M. Gurtin, R. MacCamy, *On the diffusion of biological populations*, Math. Biosci., 33 (1977), 35-49.

[H] G. Hong, *The singular homogeneous solutions to one phase free boundary problem*, Proc. Amer. Math. Soc., 143 (2015), no. 9, 4009-4015.

[IS] Imbert C., Silvestre L., *Estimates on elliptic equations that hold only where the gradient is large*, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 6, 1321–1338.

[JS] D. Jerison, O. Savin, *Some remarks on stability on cones for the one-phase free boundary problem*, Geom. Funct. Anal., 25 (2015), no. 4, 1240-1257.

[P] D. Phillips, *A minimization problem and the regularity of solutions in the presence of a free boundary*, Indiana Univ. Math. J., 32 (1983), no. 1, 1-17.

[SY1] Savin O., Yu H., *Concentration of cones in the Alt-Phillips problem* arXiv:2503.03626

[SY2] Savin O., Yu H., *Stable and Minimizing Cones in the Alt-Phillips Problem* arXiv:2502.18192

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