Symmetry and uniqueness via a variational approach

Talk 1: applications to aggregation-diffusion equation

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The Keller-Segel equation

- The Keller-Segel equation models the collective motion of cells attracted by a self-emitted chemical substance. The parabolic-elliptic Keller-Segel equation in 2D is

$$\rho_t = \Delta\rho + \nabla \cdot (\rho \nabla(\mathcal{N} * \rho)),$$

  where $\mathcal{N} = \frac{1}{2\pi} \log |x|$ is the Newtonian potential in $\mathbb{R}^2$.

  (Patlak '53, Keller-Segel '71)

- There exists a "critical mass" $M_c = 8\pi$ such that:
  - $M > 8\pi$: All solutions with initially bounded second moment must blow up in finite time.

    (Biler-Nadzieja '94, Nagai '01, Blanchet-Dolbeault-Perthame '06, Collot-Ghoul-Masmoudi-Nguyen '21)
  - $M < 8\pi$: Solutions remain bounded globally in time, and converge to a self-similar solution with the heat equation scaling.

    (Jager-Luckhaus '92, Dolbeault-Perthame '04, Blanchet-Dolbeault-Perthame '06)
  - $M = 8\pi$: No blow-up, but solutions with initially bounded second moment will aggregate as $t \to \infty$.

    (Biler-Karch-Laurençot-Nadzieja '06, Blanchet-Carrillo-Masmoudi '08, Blanchet-Carlen-Carrillo '12, Carlen-Figalli '13)

Aggregation equation with (degenerate) diffusion

- In this talk, we consider

$$\rho_t = \underbrace{\Delta \rho^m}_{\text{local repulsion}} + \underbrace{\nabla \cdot (\rho \nabla (W * \rho))}_{\text{nonlocal interaction}} \quad \text{in } \mathbb{R}^d,$$

  where $m \geq 1$, W is radially symmetric, and $W(r)$ is increasing. (So W is an attractive interaction potential).

- The nonlinear diffusion term with $m > 1$ models the anti-overcrowding effect. (Boi-Capasso-Morale '00, Topaz-Bertozzi-Lewis '06)
- The global well-posedness v.s. blow-up criteria has been well studied. (e.g. If $W = \mathcal{N}$, then $m > 2 - \frac{2}{d}$ leads to global existence, whereas solution may blow-up if $m < 2 - \frac{2}{d}$.) (Blanchet-Carrillo-Laurencot '09, Bedrossian-Rodriguez-Bertozzi '11)
- In the cases that well-posedness is known, long time behavior of solution remains unclear.

Free energy functional

- The associated free energy functional plays an important role:

$$E[\rho] = \underbrace{\frac{1}{m-1}\int \rho^m dx}_{=:S[\rho]\ \text{(entropy)}} + \underbrace{\frac{1}{2}\int \rho(\rho * W)dx}_{=:I[\rho]\ \text{(interaction energy)}} .$$

(When $m = 1$, the first term becomes $\int \rho \log \rho dx$).

- Formally taking time derivatives along a solution, we have

$$\frac{d}{dt}E[\rho] = -\int \rho \left|\nabla(\frac{m}{m-1}\rho^{m-1} + \rho * W)\right|^2 dx \leq 0.$$

- Formally, the solution is a gradient flow of E in the metric space endowed by the 2-Wasserstein distance. (But rigorously justifying this requires certain convexity of W).
(Villani'03, Ambrosio-Gigli-Savare '08, Craig '17)

Scaling argument for the free energy

Using a scaling argument, one can *formally* see whether there should be global existence or finite-time blow-up.

Below is the argument for Newtonian potential (one can also apply it to a general power-law kernel).

$$E[\rho] = \underbrace{\frac{1}{m-1}\int \rho^m dx}_{=:S[\rho] \ \text{(entropy)}} + \underbrace{\frac{1}{2}\int \rho(\rho * \mathcal{N})dx}_{=:I[\rho] \ \text{(interaction energy)}} .$$

As we replace ρ by $\rho_\lambda := \lambda^d \rho(\lambda x)$:

- $S[\rho_\lambda] = \lambda^{(m-1)d} S[\rho]$.
- Since $\mathcal{N}(x) = c_d |x|^{2-d}$, we have $I[\rho_\lambda] = \lambda^{d-2} I[\rho]$.

Thus the equation has the following three regimes:

- $m = m_c := 2 - 2/d$: fair competition case. (Note that when $d = 2$, we have $m_c = 1$).
- $m > m_c$: diffusion-dominated when $\lambda \gg 1$.
- $m < m_c$: aggregation-dominated when $\lambda \gg 1$.

Well-posedness and dynamics of solution

The above formal argument can be made rigorous using the HLS inequality

$$\int \rho(\rho * \mathcal{N})dx \le C_d M^{2/d} \int \rho^{m_c} dx,$$

and it also yields a critical mass M_c in the fair competition case.

For each regime, the following is known:

- $m > m_c$: for any $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, solution exists globally in time, and the L^∞ norm stays uniformly bounded in time.
  (Sugiyama '06, Carrillo-Calvez '06)
  Open question: Long time behavior of solutions remain unclear. (Will discuss in more details later.)
- $m < m_c$: For any $M > 0$, there exists solutions that blow-up in finite time. Meanwhile, solutions with sufficiently small initial data dissipate with the porous medium equation scaling.
  (Sugiyama '06, Bedrossian '11, Bian-Liu '13, Chen-Liu-Wang '14)

- $m = m_c$: there is a critical mass M_c (depending only on d), such that:
  - If $M < M_c$, solutions are bounded globally in time, and there exists self-similar solutions that dissipate with the porous medium equation scaling.
    (Blanchet-Carrillo-Laurençot '09, Bedrossian '11)
  - If $M > M_c$, all radial solutions blow up in finite time.
    (Bedrossian-Kim '13)

    Open question: Must all non-radial solutions blow-up in finite time too?
  - If $M = M_c$, there is a family of compactly supported stationary solutions that are scalings of each other. Every radial solution with compact support will converge to some stationary solution.
    (Blanchet-Carrillo-Laurençot '09, Y. '14)

    Open question: What about dynamics for non-radial solutions with mass $M = M_c$?

Main questions

Below we focus on the diffusion-dominated regime where blow-up doesn't happen. In order to understand the long-time dynamics, a key step is to identify the stationary solutions.

Question

1. *For a given mass, does there exist a stationary solution?*
2. *Are they necessarily radially symmetric (up to a translation)?*
3. *If so, is it unique within the radial class?*
4. *If yes, is it the global attractor of the dynamics?*

Question 1 already have a satisfactory answer:

- For a given mass, the global minimizer of $E[\rho] = \frac{1}{m-1}\int \rho^m dx + \frac{1}{2}\int \rho(\rho * W)dx$ is a stationary solution.
- Existence can be obtained by a concentration-compactness argument (Lieb–Oxford '81, Lions '84, Bedrossian '11, Carrillo–Delgadino–Patacchini '18).
- By Riesz rearrangement inequality, the *global minimizer* of E must be radially decreasing.

Thank you for your attention!