# Symmetry and uniqueness via a variational approach

## Talk 2: applications to aggregation-diffusion equation (cont.)

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# Aggregation equation with (degenerate) diffusion

- Consider the aggregation-diffusion equation

$$\rho_t = \underbrace{\Delta \rho^m}_{\text{local repulsion}} + \underbrace{\nabla \cdot (\rho \nabla (W * \rho))}_{\text{nonlocal interaction}} \quad \text{in } \mathbb{R}^d,$$

where $m \geq 1$, $W$ is radially symmetric, and $W(r)$ is increasing. (So $W$ is an attractive interaction potential).

- The associated free energy functional plays an important role:

$$E[\rho] = \underbrace{\frac{1}{m-1} \int \rho^m dx}_{=: S[\rho] \text{ (entropy)}} + \underbrace{\frac{1}{2} \int \rho(\rho * W) dx}_{=: I[\rho] \text{ (interaction energy)}}.$$

(When $m = 1$, the first term becomes $\int \rho \log \rho dx$).

- Last time: all global minimizers are radially decreasing up to a translation.
- Today: What about critical points?

# Symmetric or not?

## Question

*Must every stationary solution be radial? It doesn't need to be a global minimizer!*

We give a positive answer for all attractive kernels $W$ that is no more singular than Newtonian (which was later generalized to more singular kernels by Carrillo-Hoffmann-Mainini-Volzone '18):

## Theorem (Carrillo-Hittmeir-Volzone-Y., '19)

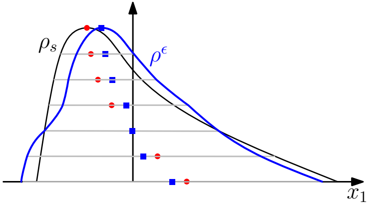
*Let $W$ be an attractive potential that is no more singular than Newtonian kernel. Any stationary solution $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ must be radially decreasing up to a translation.*

Key idea: If $\rho_s$ is non-radial up to a translation, it can't be a critical point of

$$E[\rho] = \frac{1}{m-1}\int \rho^m dx + \frac{1}{2}\int \rho(\rho * W)dx.$$

# Sketch of the symmetry proof

- Assuming a stationary solution $\rho_s$ is non-radial, we perturb it using its continuous Steiner symmetrization:




- Since $\int \rho_s^m = \int (\rho^\epsilon)^m$, and interaction energy decreases in the first order for a short time (need some work to check this!),

$$E_m[\rho^\epsilon] - E_m[\rho_s] < -c\epsilon \quad \text{for all sufficiently small } \epsilon > 0,$$

  where $c > 0$ depending on $\rho_s$ and $W$.
- If the equation has a rigorous gradient flow formulation, the above argument implies that $|\partial E|[\rho_s] \geq c$, directly leading to a contradiction.
- If there is no rigorous gradient flow formulation (which is the case for most kernels with singularity at origin), we need to manually derive a contradiction and it gets a lot more technical!

# Unique or not?

## Question

*For attractive kernels, for a given mass, are steady states unique?*

Uniqueness results are only known in the following cases:

- For Newtonian potentials, in the diffusion-dominated regime: (Lieb–Yau '87, Kim–Yao '12, Carrillo–Castorina–Volzone '15)
- For Riesz potentials, in the diffusion-dominated regime: (Carrillo–Hoffmann–Mainini–Volzone '18, Calvez–Carrillo–Hoffmann '19, Chan–Gonzalez–Huang–Mainini–Volzone '20)
- For convex potential $W$ (McCann '97)
- For the special power $m = 2$ and $W$ is a $C^2$ attractive potential. (Burger–Di Francesco–Franek '13, Kaib '17)

# Uniqueness for $m \geq 2$

**Theorem (Delgadino–Yan–Y., '22)**

*Let $m \geq 2$ and $W \in C^1(\mathbb{R}^d \setminus \{0\})$ be an attractive potential with $W'(r) \lesssim r^{-d-1+\delta}$ for some $\delta > 0$ for all $r \in (0,1)$. Then there is at most one steady state (up to a translation) for any given mass .*

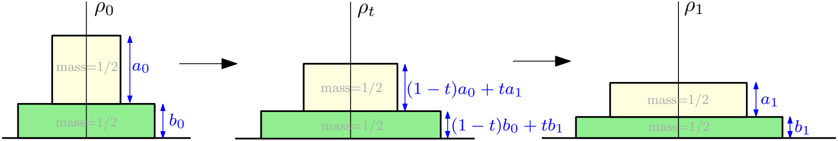
Idea of proof (when the gradient flow structure is rigorous):

- Stationary solutions are critical points of the energy functional.
- For $m \geq 2$, if $\rho_0$, $\rho_1$ are two radial stationary solutions with the same mass, we will construct a curve $\{\rho_t\}_{t=0}^1$ connecting them, such that the energy along this curve is strictly convex.
- Therefore $\rho_0$ and $\rho_1$ can't be both critical points!

But how to find such an interpolation curve?
(Note: linear interpolation or 2-Wasserstein geodesic do not work!)

# Construction of the interpolation curve

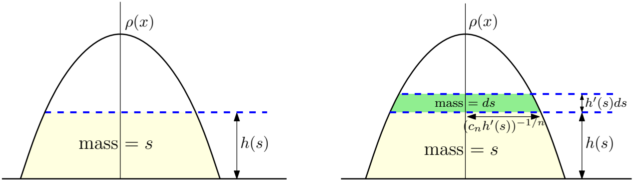
- Suppose $\rho_0$, $\rho_1$ are two radially decreasing step functions having $N$ horizontal layers with mass $1/N$ in each layer.
- $\rho_t$ is constructed by deforming each layer so that its height changes linearly, and meanwhile adjust the width so that the mass in each layer remains constant.




- Note that $\rho_t$ is neither the linear interpolation between $\rho_0$ and $\rho_1$, nor the geodesic in 2-Wasserstein metric.
- For two radially decreasing function, the interpolation can be seen as a $N \to \infty$ limit of the step-function case.

# Construction of the interpolation curve

- For a radially decreasing function $\rho$ with mass 1, define its "height function with respect to mass" $h(s)$ as the left figure:




- $\rho$ can be uniquely recovered from $h$ (see the right figure):

$$\rho(x) = \int_0^1 1_{B(0,(c_d h'(s))^{-1/d})}(x) h'(s) ds$$

- Let $h_0, h_1$ be the height function for $\rho_0, \rho_1$. For $t \in (0,1)$, let

$$h_t(s) = (1-t)h_0(s) + th_1(s),$$

and let $\rho_t$ be determined by the height function $h_t$.

# Convexity of energy

- For the entropy, an explicit computation gives

$$S[\rho] = \int_{\mathbb{R}^d} \frac{1}{m-1}\rho^m dx$$
$$= \int_0^{\max\rho} \frac{m}{m-1} h^{m-1}|\{\rho > h\}|dh$$
$$= \int_0^1 \frac{m}{m-1} h(s)^{m-1} ds,$$

thus
$$\frac{d^2}{dt^2}S[\rho_t] = m(m-2)\int_0^1 (h_1 - h_0)^2 h_t(s)^{m-3} ds,$$

which is non-negative if and only if $m \geq 2$.

- The interaction energy $I[\rho] = \int \rho(\rho * W)dx$ is strictly convex along the curve for all attractive potential $W$, but the proof is more technical.

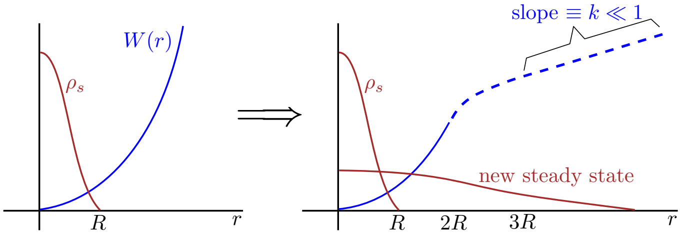
# Non-uniqueness for $1 < m < 2$

For all $m < 2$, our uniqueness proof fails. But is there really non-uniqueness in this regime?

**Theorem (Delgadino–Yan–Y., '22)**

*Let $1 < m < 2$. There exists a smooth attractive kernel $W$ which gives an infinite sequence of radially decreasing stationary solutions with the same mass.*

- It shows that the uniqueness result for $m \geq 2$ is indeed sharp.




- Claim: If $k > 0$ is sufficiently small, then it leads to a different stationary solution from $\rho_s$.

# Scaling argument for integrable kernels

Reason: If $k = 0$, then $W$ becomes an integrable attractive kernel. For such kernel, a heuristic scaling argument shows a sufficiently flat initial data should continue spreading for $1 \leq m < 2$:

As we replace $\rho$ by $\rho_\lambda := \lambda^d \rho(\lambda x)$, the entropy and interaction energy scales as follows as $\lambda \to 0$:

$$S[\rho_\lambda] = \lambda^{(m-1)d} S[\rho] = \frac{\lambda^{(m-1)d}}{m-1} \int \rho^m dx,$$

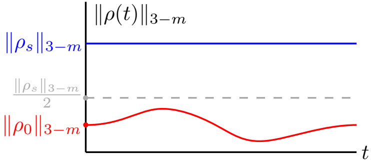
$$I[\rho_\lambda] \to \frac{\lambda^d}{2} \|W\|_{L^1} \int \rho^2 dx + o(\lambda^d).$$

Thus we formally expect the following:

- $m = 2$ (critical power): here both terms scale the same as $\lambda \to 0$.
- $1 \leq m < 2$: $E[\rho_\lambda] > 0$ for sufficiently small $\lambda > 0$.
  (i.e. It is energy favorable for a sufficiently flat initial data to spread more.)
- $m > 2$: $E[\rho_\lambda] < 0$ for sufficiently small $\lambda > 0$.

# Leading to non-uniqueness

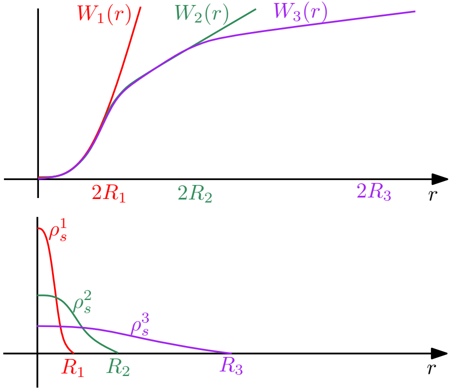
- Let $1 < m < 2$. To rigorously justify the heuristics, we use a standard energy estimate to track the evolution of $L^{3-m}$ norm of a solution.
- We show if $0 < k \ll 1$ and $\|\rho_0\|_{3-m} \leq \frac{\|\rho_s\|_{3-m}}{2}$, then $\|\rho(t)\|_{3-m}$ is bounded by $\frac{\|\rho_s\|_{3-m}}{2}$ for all times, so $\rho(t)$ can never return to $\rho_s$.




- But $\{\rho(t)\}_{t>0}$ must remain tight, since $W(r) \sim kr$ for $r \gg 1$, implying the first moment of $\rho(t)$ is uniformly bounded in time.
- Uniform-in-time $L^{3-m}$ bounds + tightness + energy dissipation $\Rightarrow$ existence of a new stationary solution.

# Infinite sequence of stationary solutions

- Finally, an iterative procedure allows us to construct a kernel with an infinite number of stationary solutions (all with the same mass, and radially decreasing).



# Open questions

**Question**

*For a given mass, are stationary solutions unique when $m = 1$?*

- They are still known to be radially decreasing, but both our uniqueness and non-uniqueness proofs fail in the $m = 1$ case.

**Question**

*When $m > 2$, does the solution converge to the unique stationary solution with the same mass and center of mass as the initial data?*

- Difficulty: need to show mass can't escape to infinity.
- Some recent progress by Ruiwen Shu '20 in 1D and 2D for large $m$.

Thank you for your attention!