# Symmetry and uniqueness via a variational approach

Day 3: application to 2D Euler equation

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# What's the shape of my planet?

Let's start with the following example (seemingly unrelated to fluids):

The little prince wants to know the shape of his planet $D \subset \mathbb{R}^n$.

Somehow, the only thing he can measure is the gravitational potential $1_D * \mathcal{N}$ at each point on the surface.

After years of hard work, he found $(1_D * \mathcal{N})(x) = \text{const on } \partial D$.

**Question:** Must $D$ be a ball?

It turns out this question is closely related to stationary vortex patches!

# 2D incompressible Euler equation

- 2D Euler equation in vorticity form:

$$\omega_t + u \cdot \nabla \omega = 0,$$

with Biot-Savart law $u = \nabla^\perp(\omega * \mathcal{N})$, where $\mathcal{N} = \frac{1}{2\pi} \log |x|$ is the Newtonian potential in $2D$.

- The global regularity for smooth initial data has long been established since works of Wolibner '33 and Hölder '33.

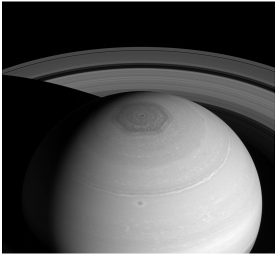
- Global well-posedness is also established for weak solution with $\omega_0 \in L^1 \cap L^\infty$: Yudovich '63, with $\omega(x,t) = \omega_0(\phi^{-1}(t,x))$.

- A patch solution has $\omega_0 = 1_D$, with $D$ being a bounded domain. We then have $\omega(x,t) = 1_{D_t}$, with $D_t = \phi(t, D)$.

- Global regularity of patches in $C^{1,\gamma}$ is first proved by Chemin '93, and a shorter proof is given by Bertozzi–Constantin '93.

# Solutions whose shape remains unchanged in time

- Instead of the dynamical solutions, our focus will be the solutions whose shape remains unchanged in time.
- These include the stationary solutions, as well as uniformly-rotating solutions.
- $D$ is a stationary patch $\iff$ $1_D * \mathcal{N} = \text{const}$ on each component of $\partial D$.
- $D$ is a uniformly-rotating patch with angular velocity $\Omega \iff$ $1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2 = \text{const}$ on each component of $\partial D$.



Rotating hexagon on the north pole of Saturn.
Source: Wikipedia

# Symmetric or not?

Simple observation: Any radially symmetric patch $D$ satisfies the equation of rotating patch for any $\Omega \in \mathbb{R}$.

## Question

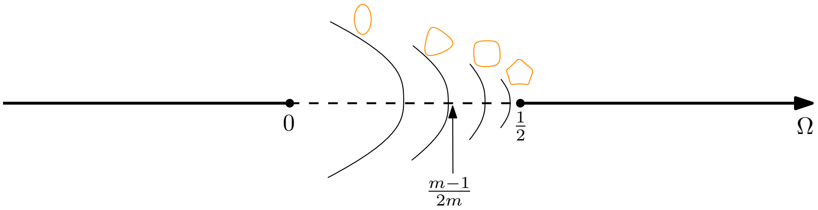
*Under what condition must a stationary/rotating patch be radially symmetric?*

Positive answer in the following cases: if $\omega_0 = 1_D$, then

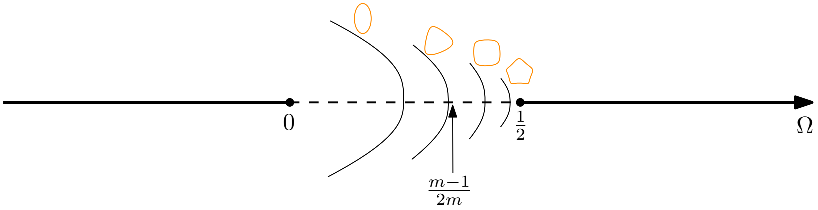
- Fraenkel '00: If $D$ is simply-connected and $\Omega = 0$, it must be a disk. Proof based on the moving plane method.
- Hmidi '14: If $D$ is convex and $\Omega < 0$, it must be a disk.
- Hmidi '14: If $D$ is simply-connected and $\Omega = 1/2$, it must be a disk.

# Non-radial uniformly rotating solutions

- Kirchhoff vortex (1876): any ellipse of semiaxis $a, b$ is a rotating patch with $\Omega = \frac{ab}{(a+b)^2}$.
- Deem–Zabusky '78: numerical evidence of rotating patches with $m$-fold symmetry.
- Burbea '82 proved that there exists a family of $m$-fold rotating patches bifurcating from the disk at $\Omega = \frac{m-1}{2m}$. ($m = 2$ gives Kirchhoff ellipses.)
- Boundary regularity: Hmidi–Mateu-Verdera'13, Castro–Córdoba–Gómez-Serrano'15
- Other perspectives: bifurcation from doubly-connected patch; pairs of patches; secondary bifurcation; smooth rotating solution; stability...



# Open question: what's at the end of the bifurcation curve?




The global bifurcation is studied in Hassainia–Masmoudi–Wheeler '17, where they showed towards the end of the bifurcation curve, there are points on the boundary whose angular fluid velocity becomes arbitrarily small.

Two intriguing open question remains for $m \geq 3$:

- Formation of $90^\circ$ angle at the end of bifurcation curve?
- Does "cat's eye" persist along the bifurcation curve?

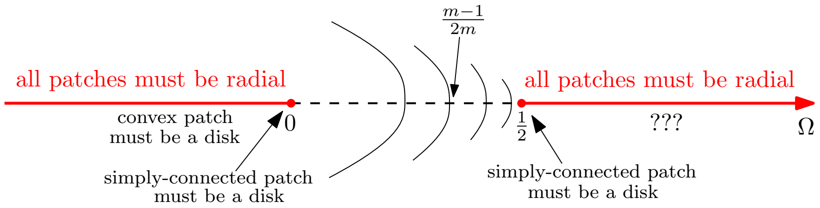
See some fascinating numerics at Miles Wheeler's page:
`https://www.mileshwheeler.com/patchnumerics.html`

# No non-trivial patch for $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$

**Theorem (Gómez-Serrano, Park, Shi, and Y., '21)**

*Let $D$ be a stationary/rotating patch (not necessarily connected or simply-connected) with angular velocity $\Omega$.*

- *If $\Omega < 0$ or $\Omega \geq 1/2$, then $D$ must be radially symmetric.*
- *And if $\Omega = 0$, then $D$ is radial up to a translation.*




- Instead of moving plane method, our proof has a calculus-of-variation flavor.

# Simply-connected patch are radial for $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$

- The proof is very short for simply-connected patch $D$. Towards a contradiction, assume $D$ is not a disk, and it is stationary/rotating with $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$.
- Idea: Consider the first variation of the "energy functional"

$$E[D] = -\int_{\mathbb{R}^2} \frac{1}{2} 1_D (1_D * \mathcal{N}) - \frac{\Omega}{2} |x|^2 1_D \, dx$$

  along a carefully chosen deformation of $D$.
- For the transport equation $\rho_t + \nabla \cdot (\rho \vec{v}) = 0$ with initial data $\rho(x, 0) = 1_D$, we have

$$\frac{d}{dt} E[\rho]\Big|_{t=0} = -\int_D \vec{v}(x) \cdot \nabla \Big( \underbrace{(1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2}_{=: f(x)} \Big) dx =: \mathcal{I}$$

- On the one hand, using $f = C$ on $\partial D$, divergence theorem gives $\mathcal{I} = 0$ for any smooth $\vec{v}$ with $\nabla \cdot \vec{v} = 0$ in $D$.

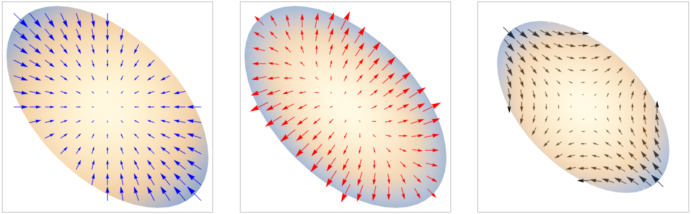
# Perturbing $D$ by a divergence-free vector field

- On the other hand, if $D$ is simply-connected and not a disk, we construct an explicit smooth $\vec{v}$ with $\nabla \cdot \vec{v} = 0$ in $D$, and show that $\mathcal{I} \neq 0$ if $\Omega \in (-\infty, 0] \cup [1/2, \infty)$.
- We define $\vec{v} : \overline{D} \to \mathbb{R}^2$ as

$$\vec{v}(x) := -\vec{x} - \nabla p,$$

where $p$ solves the Poisson equation $\begin{cases} \Delta p = -2 & \text{in } D, \\ p = 0 & \text{on } \partial D. \end{cases}$

- Note that $\nabla \cdot \vec{v} = 0$ in $D$.



$-\vec{x}$ $\quad$ $-\nabla p$ $\quad$ $\vec{v} = -\vec{x} - \nabla p$

# Obtaining a contradiction for $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$

- For such $v$, an explicit computation gives

$$\begin{aligned}\mathcal{I} &= \int_D x \cdot \nabla\left(1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx + \int_D \nabla p \cdot \nabla f dx \\ &= \frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 dx + (2\Omega - 1)\int_D p dx\end{aligned}$$

- For $|D|$ fixed, $\int_D |x|^2 dx$ is minimized if and only if $D$ is a disk.
- Talenti '76: If $p$ solves $\Delta p = -2$ in $D$ with $p = 0$ on $\partial D$, we have

$$\int_D p\, dx \leq \frac{1}{4\pi}|D|^2,$$

  with "$=$" achieved if and only if $D$ is a disk.

- Combining them, we have $\mathcal{I} \geq 0$ if $\Omega \leq 0$, $\mathcal{I} \leq 0$ if $\Omega \geq \frac{1}{2}$, with "$=$" achieved if and only if $D$ is a disk.

Thank you for your attention!