Symmetry and uniqueness via a variational approach

Day 4: application to 2D Euler & SQG equation

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Where we are so far

- Recall: For 2D Euler, a rotating patch with angular velocity Ω satisfies

$$f(x) := 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2 = \text{const}$$

on each connected component of ∂D (different constants on different pieces).

- Taking first variation of the "energy functional"

$$E[D] = -\int_{\mathbb{R}^2} \frac{1}{2} 1_D (1_D * \mathcal{N}) - \frac{\Omega}{2}|x|^2 1_D \, dx$$

along any divergence-free vector field $\vec{v}$ in D, we have

$$\frac{d}{dt} E[\rho]\Big|_{t=0} = -\int_D \vec{v}(x) \cdot \nabla \Big(\underbrace{(1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2}_{=: f(x)} \Big) dx =: \mathcal{I}$$

- For simply-connected D, using $f = C$ on ∂D, divergence theorem gives $\mathcal{I} = 0$.

Perturbing D by a divergence-free vector field

- On the other hand, if D is simply-connected and not a disk, we construct an explicit smooth $\vec{v}$ with $\nabla \cdot \vec{v} = 0$ in D, and show that $\mathcal{I} \neq 0$ if $\Omega \in (-\infty, 0] \cup [1/2, \infty)$.
- We define $\vec{v} : \overline{D} \to \mathbb{R}^2$ as

$$\vec{v}(x) := -\vec{x} - \nabla p,$$

where p solves the Poisson equation

$$\begin{cases} \Delta p = -2 & \text{in } D, \\ p = 0 & \text{on } \partial D. \end{cases}$$

- Note that $\nabla \cdot \vec{v} = 0$ in D.

Obtaining a contradiction for $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$

- For such v, an explicit computation gives

$$\mathcal{I} = \int_D x \cdot \nabla(1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2)dx + \int_D \nabla p \cdot \nabla f dx$$
$$= \frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 dx + (2\Omega - 1)\int_D p dx$$

- For $|D|$ fixed, $\int_D |x|^2 dx$ is minimized if and only if D is a disk.
- Talenti '76: If p solves $\Delta p = -2$ in D with $p = 0$ on ∂D, we have

$$\int_D p\, dx \le \frac{1}{4\pi}|D|^2,$$

with "$=$" achieved if and only if D is a disk.

- Let's prove Talenti's rearrangement theorem! (on the board)

- Combining them, we have $\mathcal{I} \ge 0$ if $\Omega \le 0$, $\mathcal{I} \le 0$ if $\Omega \ge \frac{1}{2}$, with "$=$" achieved if and only if D is a disk.

Dealing with non-simply-connected patches

- If D is not simply-connected,

$$f = \mathcal{N} * \omega - \frac{\Omega}{2}|x|^2 = C_i \text{ on } \partial D_i, \qquad \text{where } C_i \text{ can be different.}$$

  So our first computation of $\mathcal{I} = 0$ no longer holds!

- To ensure $\mathcal{I} = 0$, we require $\vec{v}$ be divergence free in D and satisfies $\int_{\partial D_i} \vec{v} \cdot n d\sigma = 0$ for each ∂D_i.

- Idea: still let $\vec{v} = -\vec{x} - \nabla p$, but modify p into $\Delta p = -2$ in D, $p = c_i$ on ∂D_i for suitable c_i. Also need to modify the proof of Talenti's theorem for such p.

- Such modification gives us that any connected patch (not necessarily simply-connected) must be radial for $\Omega \leq 0$ or $\Omega \geq 1/2$.

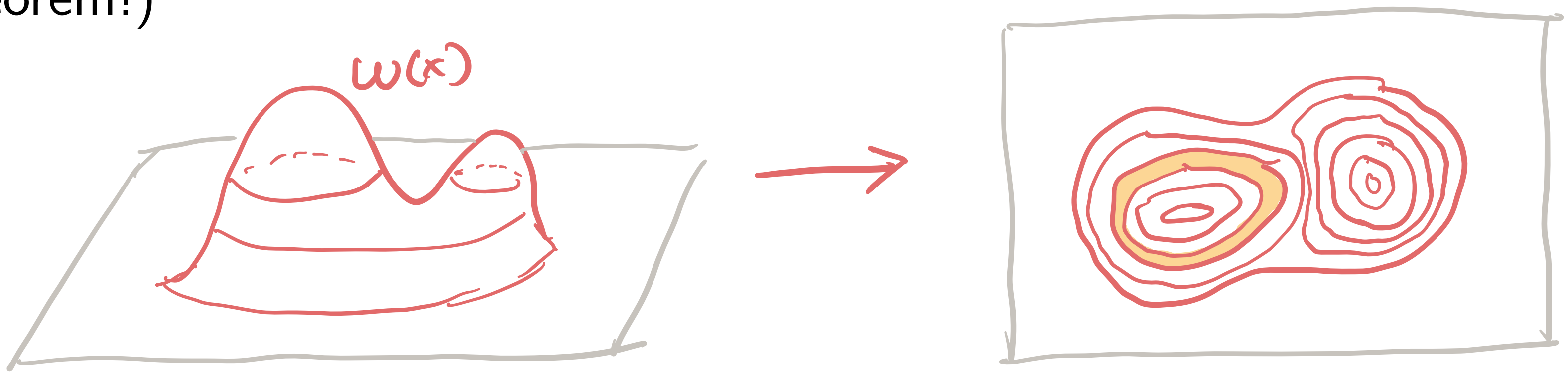
Stationary patch/smooth solution

For smooth stationary solutions we can also say the following:

Theorem (Gómez-Serrano, Park, Shi, and Y., '21)

Assume ω is a smooth stationary solution with compact support (or fast decay at infinity). If ω does not change sign, it must be radial up to a translation.

- Idea of proof: approximate a smooth ω by step functions, then apply the previous perturbation for each layer. (Need a quantitative version of Talenti's theorem!)



- Note: If vorticity is allowed to change sign, one can construct nonradial compactly-supported stationary solutions.

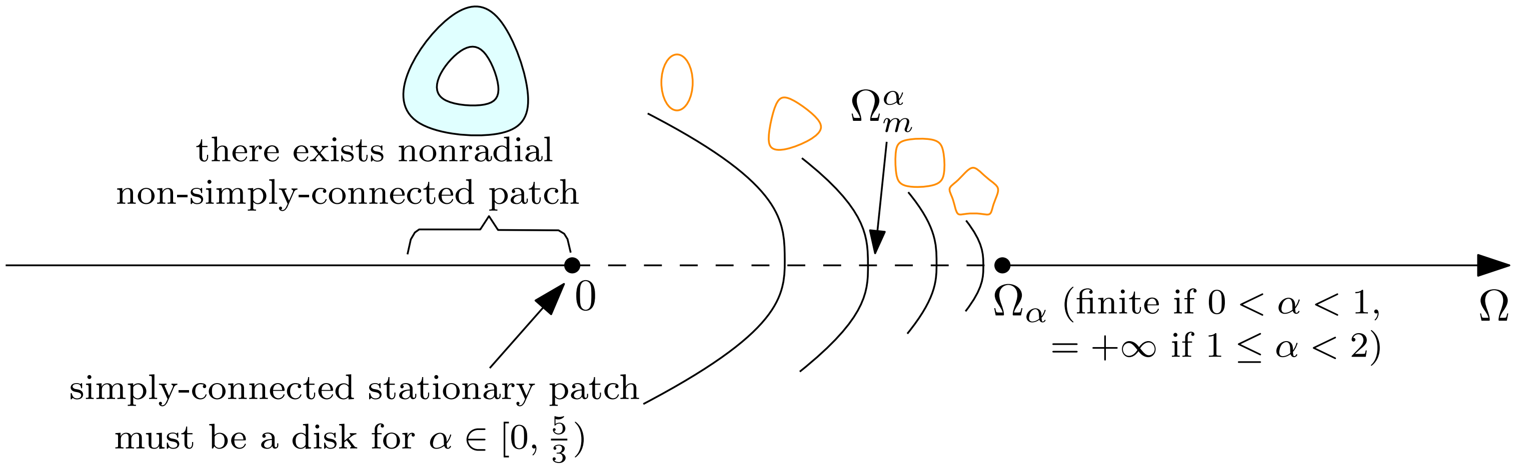
  (Gómez-Serrano–Park–Shi '21, Enciso–Fernández–Ruiz–Sicbaldi '23).

SQG and generalized SQG

- Consider the Biot-Savart law $u = \nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\omega = \nabla^\perp(\mathcal{K}_\alpha * \omega)$, for $\alpha \in (0, 2)$. ($\alpha = 0 \Rightarrow$ 2D Euler; $\alpha = 1 \Rightarrow$ SQG)
- A rotating patch D with angular velocity Ω satisfies

$$1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 = \text{const} \quad \text{on } \partial D.$$

- Existence of patch/smooth rotating solution (for some $\Omega > 0$) given by Castro–Córdoba–Gómez-Serrano '16.
- For $0 < \alpha < 5/3$, all *simply connected* stationary patches are disks. (Reichel '09, Lu–Zhu '12, Choksi–Neumayer–Topaloglu '18, moving plane method).
- Non-simply-connected stationary patches are not necessarily radial: For $\alpha \in (0, 2)$, Gómez-Serrano '18 showed there exists non-radial stationary patches bifurcating from an annulus.

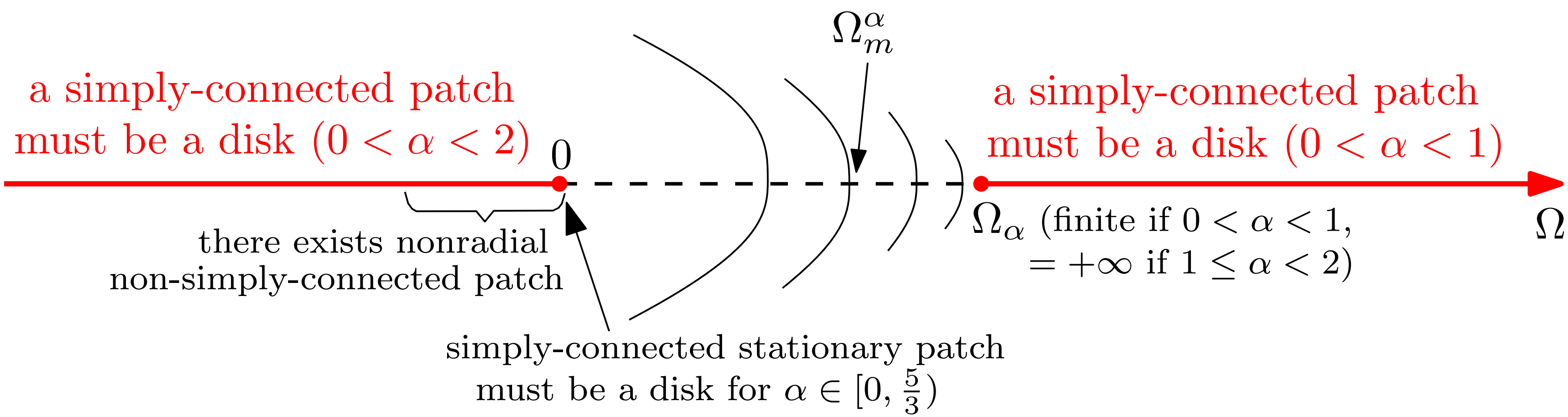


Symmetry of stationary/rotating patches

Theorem (Gómez-Serrano, Park, Shi, and Y., '21)

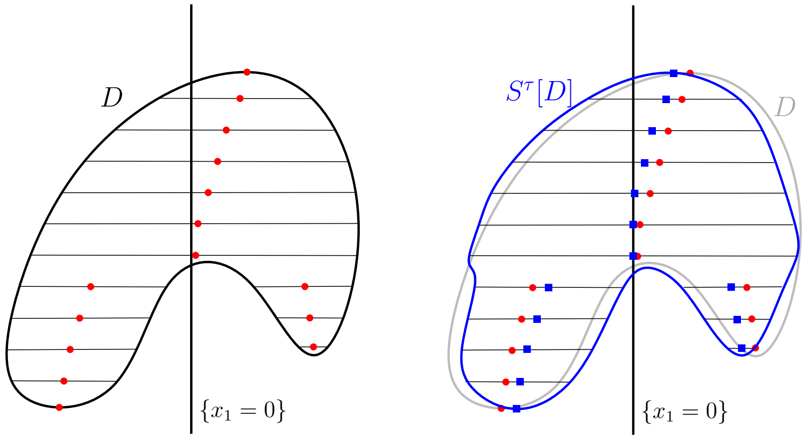
Let D be a simply-connected rotating patch to the gSQG equation with angular velocity Ω. Then:

- *For $\alpha \in (0,2)$, if $\Omega \leq 0$, the patch must be a disk.*
- *For $\alpha \in (0,1)$, there exists a constant $\Omega_\alpha = 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2}\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}$ (sharp and explicit) such that if $\Omega \geq \Omega_\alpha$ the patch must be a disk.*



Symmetry for $\Omega \leq 0$ case

- Known: $1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 = \text{const on } \partial D$.
- Let $E[D] := \frac{1}{2}\int 1_D(1_D * \mathcal{K}_\alpha) - \frac{\Omega}{2}|x|^2 dx$.
- Let us perturb D by continuous Steiner symmetrization, in a similar spirit as Carrillo–Hittmeir–Volzone–Y. '19.



- Under this perturbation, $E[D]$ decreases to the first order of τ, i.e. $E[S^\tau[D]] - E[D] < -c\tau$.

- But using that $1_D * \mathcal{K}_\alpha - \frac{1}{2}\Omega|x|^2 = C$ on ∂D, we also have $E[S^\tau[D]] - E[D] = o(\tau)$, a contradiction.

The overall story

- If some steady state / rotating solutions are critical points of some associated energy functional, one way to show symmetry/uniqueness is to build your own purturbation / interpolation to decrease the energy by the first order.
- There's no universal method to build it though! Trials and errors are needed.

Disclaimer: only successful examples are shown; failed attempts are hidden on purpose :)

Thank you for your attention!